UNIVERSITI SAINS MALAYSIA

First Semester Examination
2014/2015 Academic Session

December 2014/January 2015

MAT 517 – Computational Linear Algebra
[Aljabar Linear Pengkomputeran]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **NINE** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Apply the Gaussian elimination algorithm without interchanges on the following linear system:

$$10^{-10}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 3$$

Is the algorithm stable? Explain.

[10 marks]

1. Gunakan algoritma penghapusan Gauss tanpa penukaran ke atas sistem linear berikut:

$$10^{-10}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 3$$

Adakah algoritma ini stabil? Terangkan.

[10 markah]

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the singular values of $\mathbf{A}$ and determine the

(a) rank of the matrix,

(b) condition number with respect to 2-norm.

[10 marks]

2. Biar

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Cari nilai singular $\mathbf{A}$ *dan tentukan*

(a) pangkat matriks tersebut,

(b) nombor syaratnya berasaskan norma-2.

[10 markah]

3. Let

$$\mathbf{A}=\begin{bmatrix}1 & -1 & 3\\ 0 & 5 & 2\\ 0 & 0 & 7\end{bmatrix}, \quad \text{and} \quad \mathbf{B}=\begin{bmatrix}1 & 0 & 0\\ 1 & 10 & 0\\ 1 & 1 & 9.8\end{bmatrix}.$$

Explain why the eigenvalue problem associated with $\mathbf{A}$ is better conditioned compared to the eigenvalue problem associated with $\mathbf{B}$.

[10 marks]

3. Biar

$$\mathbf{A}=\begin{bmatrix}1 & -1 & 3\\ 0 & 5 & 2\\ 0 & 0 & 7\end{bmatrix}, \quad \textit{dan} \quad \mathbf{B}=\begin{bmatrix}1 & 0 & 0\\ 1 & 10 & 0\\ 1 & 1 & 9.8\end{bmatrix}.$$

Terangkan mengapa masalah nilai eigen yang berkaitan dengan $\mathbf{A}$ *adalah lebih bersyarat baik berbanding masalah nilai eigen yang berkaitan dengan* $\mathbf{B}$.

[10 markah]

4. Determine the number of floating point operations (flops) for the computation of the matrix $\mathbf{A}=\dfrac{\mathbf{u}\mathbf{v}^T}{\mathbf{u}^T\mathbf{v}}$, where $\mathbf{u}$ and $\mathbf{v}$ are column vectors of length m

[10 marks]

4. Tentukan bilangan operasi titik apungan (flops) untuk mengira matriks $\mathbf{A}=\dfrac{\mathbf{u}\mathbf{v}^T}{\mathbf{u}^T\mathbf{v}}$, *di mana* $\mathbf{u}$ *dan* $\mathbf{v}$ *adalah vektor lajur yang panjangnya* m

[10 markah]

5. Write down the Householder matrix that will zero out the last 3 entries of the following vector:

$$\begin{bmatrix} -5 \\ 4 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

[20 marks]

5. Tuliskan matriks Householder yang akan mensifarkan 3 pemasukkan terakhir vektor berikut:

$$\begin{bmatrix} -5 \\ 4 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

[20 markah]

6. Determine the FULL singular value decomposition of

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[20 marks]

6. Tentukan penghuraian nilai singular PENUH bagi

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[20 markah]

7. The modified Gram-Schmidt algorithm for computing the reduced QR factorization of an $m \times n$ matrix $\mathbf{A}$ is given below:

Algorithm I

Input: $\mathbf{A} = \left(\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)} \quad \cdots \quad \mathbf{a}^{(n)}\right) \in R^{m\times n}$, $\quad rank(\mathbf{A}) = n$

Output: Reduced QR factorization $\mathbf{A} = \mathbf{QR}$;

$$\mathbf{Q} = \left(\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n\right) \in R^{m\times n}$$

$$\mathbf{R} = \left\{r_{ij}\right\}_{i,j=1}^{n} \in R^{n\times n}$$

Set $\mathbf{q}_k = \mathbf{a}^{(k)}$, for $k = 1, 2, \ldots, n$

For $k = 1, 2, \ldots, n$

$r_{kk} = \left\|\mathbf{q}_k\right\|_2$

$\mathbf{q}_k = \dfrac{\mathbf{q}_k}{r_{kk}}$

  For $i = k+1, \ldots, n$

  $r_{ki} = \mathbf{q}_k^T \mathbf{q}_i$

   $\mathbf{q}_i = \mathbf{q}_i - r_{ki}\mathbf{q}_k$

  End

End

Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. Use Algorithm I to find the reduced QR factorization of $\mathbf{A}$.

[20 marks]

7. *Algorithm Gram-Schmidt diubahsuai untuk pemfaktoran QR terkurangkan bagi suatu matriks* $m \times n$ $\mathbf{A}$ *diberikan di bawah:*

Algoritma I
Input: $\mathbf{A} = \left(\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)} \quad \cdots \quad \mathbf{a}^{(n)}\right) \in R^{m\times n}$, $pangkat(\mathbf{A}) = n$ *Output: Pemfaktoran QR kurang* $\mathbf{A} = \mathbf{QR}$; $\mathbf{Q} = \left(\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n\right) \in R^{m\times n}$ $\mathbf{R} = \left\{r_{ij}\right\}_{i,j=1}^{n} \in R^{n\times n}$ *Set* $\mathbf{q}_k = \mathbf{a}^{(k)}$, *untuk* $k = 1, 2, \ldots, n$ *For* $k = 1, 2, \ldots, n$ $r_{kk} = \left\\|\mathbf{q}_k\right\\|_2$ $\mathbf{q}_k = \dfrac{\mathbf{q}_k}{r_{kk}}$ *For* $i = k+1, \ldots, n$ $r_{ki} = \mathbf{q}_k^T \mathbf{q}_i$ $\mathbf{q}_i = \mathbf{q}_i - r_{ki}\mathbf{q}_k$ *End* *End*

Biar $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. *Guna Algoritma I untuk mencari pemfaktoran QR terkurangkan bagi* $\mathbf{A}$.

[20 markah]

8. Consider the problem of minimizing $\|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2^2$.

(a) Let the reduced QR factorization of $\mathbf{A} \in R^{m\times n}$ be given by $\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1$, where $\mathbf{Q}_1 \in R^{m\times n}$ has orthogonal columns and $\mathbf{R}_1 \in R^{n\times n}$ is upper triangular. Show that

$$\|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2^2 = \|\mathbf{c}_1 - \mathbf{R}_1\mathbf{x}\|_2^2 + \|\mathbf{c}_2\|_2^2, \qquad (1)$$

where $\mathbf{c}_1 \in R^{n\times 1}$ is the first n entries of vector $\mathbf{Q}_1^T\mathbf{b}$ and $\mathbf{c}_2 \in R^{(m-n)\times 1}$ is the last $m-n$ entries of vector $\mathbf{Q}_1^T\mathbf{b}$.

(b) Assuming that the reduced QR factorization of $\mathbf{A}$ exists, use the results in (1) to write down an algorithm for computing the least squares solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, based on the QR method.

[20 marks]

8. Pertimbangkan masalah meminimakan $\|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2^2$.

(a) Biar pemfaktoran QR terkurangkan bagi $\mathbf{A} \in R^{m\times n}$ *diberikan oleh* $\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1$*, di mana* $\mathbf{Q}_1 \in R^{m\times n}$ *mempunyai lajur berortogon dan* $\mathbf{R}_1 \in R^{n\times n}$ *ialah bersegitiga atas. Tunjukkan bahawa*

$$\|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2^2 = \|\mathbf{c}_1 - \mathbf{R}_1\mathbf{x}\|_2^2 + \|\mathbf{c}_2\|_2^2, \qquad (1)$$

di mana $\mathbf{c}_1 \in R^{n\times 1}$ *ialah* n *pemasukkan pertama vektor* $\mathbf{Q}_1^T\mathbf{b}$ *dan* $\mathbf{c}_2 \in R^{(m-n)\times 1}$ *ialah* $m-n$ *pemasukkan terakhir vektor* $\mathbf{Q}_1^T\mathbf{b}$.

(b) Dengan andaian pemfaktoran QR terkurangkan bagi $\mathbf{A}$ *wujud, guna keputusan dalam (1) untuk menuliskan satu algoritma untuk mengira penyelesaian kuasa dua terkecil bagi* $\mathbf{A}\mathbf{x} = \mathbf{b}$*, berdasarkan kaedah QR.*

[20 markah]

9. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 2.9 \end{pmatrix}.$$

(a) Let $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ be the sequence of the approximation of the eigenvector of $\mathbf{A}$ (associated with the largest eigenvalue) obtained using the Power method. Starting with $\mathbf{x}^{(0)} = [0.5, 0.5]^T$, find $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and an estimate for the largest eigenvalue after 2 steps.

(b) Apply a shift of $\sigma = 2.8$ and perform 2 steps of the Power method on the shifted matrix $\mathbf{A} - \sigma\mathbf{I}$ starting with $\mathbf{x}^{(0)} = [0.5, 0.5]^T$. Compute an estimate for the largest eigenvalue of $\mathbf{A}$ and compare with the results you obtain in part a). Which method gives you a better estimate? Explain your observation.

[30 marks]

9. Pertimbangkan matriks

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 2.9 \end{pmatrix}$$

(a) Biar $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ menjadi jujukan penghampiran eigenvektor $\mathbf{A}$ (yang berkaitan dengan nilai eigen terbesar) yang diperolehi menggunakan kaedah Power. Bermula dengan $\mathbf{x}^{(0)} = [0.5, 0.5]^T$, cari $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ dan anggarkan nilai eigen terbesar selepas 2 langkah.

(b) Guna anjakan $\sigma = 2.8$ dan laksanakan 2 langkah kaedah Power pada matriks teranjakkan $\mathbf{A} - \sigma\mathbf{I}$ bermula dengan $\mathbf{x}^{(0)} = [0.5, 0.5]^T$. Kira anggaran bagi nilai eigen terbesar bagi $\mathbf{A}$ dan bandingkan dengan keputusan yanganda perolehi bagi bahagian a). Kaedah mana memberikan anda anggaran yang lebih baik? Terangkan pemerhatian anda.

[30 markah]

10. Let

$$\mathbf{A}=\begin{pmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{pmatrix}.$$

Use partial pivoting to factor $\mathbf{A}$ into the form $\mathbf{A}=\mathbf{P}^T\mathbf{LU}$ where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is a unit lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix. Use this factorization to solve the linear equation:

$$\begin{pmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\2\\-1\\2\end{pmatrix}.$$

[30 marks]

10. Biar

$$\mathbf{A}=\begin{pmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{pmatrix}.$$

Guna pangsian separa untuk memfaktorkan $\mathbf{A}$ *kepada bentuk* $\mathbf{A}=\mathbf{P}^T\mathbf{LU}$ *di mana* $\mathbf{P}$ *ialah matriks permutasi,* $\mathbf{L}$ *ialah matriks segitiga bawah unit dan* $\mathbf{U}$ *ialah matriks segitiga atas. Guna pemfaktoran ini untuk menyelesaikan persamaan linear:*

$$\begin{pmatrix}2&1&1&0\\4&3&3&1\\8&7&9&5\\6&7&9&8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\2\\-1\\2\end{pmatrix}.$$

[30 markah]

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