UNIVERSITI SAINS MALAYSIA

First Semester Examination
2014/2015 Academic Session

December 2014/January 2015

MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Suppose E, F and G are events in a sample space and $E \neq 0, F \neq 0, G \neq 0$. State whether the following statements are true or false. Justify your answers.

   (a) If E and F are mutually exclusive, then they are independent.
   (b) If $E^c \subset F$, then $F^c \subset E^c$.
   (c) If E, F and G are mutually independent, then $\overline{E}$ and $\overline{F} \cap \overline{G}$ are also mutually independent

[20 marks]

1. Andaikan E, F and G adalah peristiwa-peristiwa dalam ruang sampel dan $E \neq 0, F \neq 0, G \neq 0$. *Nyatakan sama ada pernyataan yang berikut benar atau salah. Tentusahkan jawapan anda.*

   (a) Jika E dan F saling eksklusif, maka E dan F adalah saling tak bersandar.
   (b) Jika $E^c \subset F$, *maka* $F^c \subset E^c$.
   (c) If E, F and G are mutually independent, then $\overline{E}$ *and* $\overline{F} \cap \overline{G}$ *are also mutually independent*

[20 markah]

2. In good condition, a machine produces defective items only 1% of the time. In broken condition, it produces defective items 40% of the time. Suppose that the probability that the machine breaks down is 0.10.

   (a) What is the probability that a randomly selected item is defective?
   (b) What is the probability that the machine is in good condition, given that a randomly selected item is defective?
   (c) Suppose a sample of 6 items were selected and 2 were found to be defective. Was the machine in good condition at the time?

[20 marks]

2. Semasa dalam keadaan baik, sebuah mesin menghasilkan barangan cacat hanya 1% daripada masa. Dalam keadaan rosak, mesin tersebut menghasilkan barangan cacat 4% daripada masa. Andaikan bahawa kebarangkalian mesin itu rosak ialah 0.10

   (a) Apakah kebarangkalian bahawa suatu barangan yang dipilih secara rawak adalah cacat?
   (b) Apakah kebarangkalian bahawa mesin tersebut dalam keadaan baik, jika diketahui bahawa suatu barangan yang dipilih secara rawak adalah cacat?
   (c) Andaikan suatu sampel 6 barangan dipilh secara rawak dan 2 daripadanya didapati cacat. Adakah mesin tersebut dalam keadaan baik ketika itu?

[20 markah]

3. Let X be a geometric random variable with parameter p.

(a) Compute $P(X > n)$.

(b) Show that X has the memoryless property i.e. for any integers n and m, with $n > m$,

$$P(X > n \mid X > m) = P(X > n - m).$$

[20 marks]

3. Andaikan X ialah suatu pembohubah rawak geometrik dengan parameter p.

(a) Hitung $P(X > n)$.

(b) Tunjukkan bahawa X mempunyai ciri tiada ingatan, iaitu bagi sebarang integer n dan m, dengan $n > m$,

$$P(X > n \mid X > m) = P(X > n - m).$$

[20 markah]

4. $X \in \{0, 1, 2\}$ and $Y \in \{1, 2\}$ are two discrete random variables. Their joint probability mass function is given as follows:

$$P(X = 0, Y = 1) = 0.1 \qquad P(X = 0, Y = 2) = 0.3$$
$$P(X = 1, Y = 1) = 0.2 \qquad P(X = 1, Y = 2) = 0.2$$
$$P(X = 2, Y = 1) = 0.1 \qquad P(X = 2, Y = 2) = 0.1$$

(a) What is the marginal probability mass function of Y ?

(b) Compute $E[XY]$

(c) Compute the covariance of X and Y .

(d) Are X and Y independent random variables? Justify your answer.

[20 marks]

4. $X \in \{0, 1, 2\}$ dan $Y \in \{1, 2\}$ adalah dua pembolehubah rawak diskret. Fungsi jisim kebarangkalian tercantum mereka diberikan seperti yang berikut:

$$P(X = 0, Y = 1) = 0.1 \qquad P(X = 0, Y = 2) = 0.3$$
$$P(X = 1, Y = 1) = 0.2 \qquad P(X = 1, Y = 2) = 0.2$$
$$P(X = 2, Y = 1) = 0.1 \qquad P(X = 2, Y = 2) = 0.1$$

(a) Apakah fungsi jisim kebarangkalian sut Y ?

(b) Hitung $E[XY]$

(c) Hitung kovarians X dan Y .

(d) Adakah X dan Y pembolehubah rawak tak bersandar? Tentusahkan jawapan anda.

[20 markah]

5. Suppose that X and Y are independent exponential random variables with parameter λ. If $Z = \dfrac{X}{X+Y}$

(a) find the distribution function, $F_Z(z)$, for $0 < z < 1$;

(b) find the density function of Z and name the distribution.

[20 marks]

5. Andaikan X dan Y adalah pembolehubah rawak exponen dengan parameter λ. Jika $Z = \dfrac{X}{X+Y}$

(a) dapatkan fungsi taburan, $F_Z(z)$, bagi $0 < z < 1$;

(b) dapatkan fungsi ketumpatan bagi Z dan nyatakan nama taburan tersebut.

[20 markah]

6. Let X and Y be two independent random variables, with respective probability density function

$$f(x) = xe^{-x} \qquad for\ x > 0$$

and $$f(y) = 2e^{-2y} \qquad for\ y > 0$$

(a) Write the joint probability density function of X and Y, $f_{X,Y}(x,y)$

(b) Compute the probability $P(X - Y > 2)$

(c) Find the distribution function $F_{X,Y}(x,y)$.

[20 marks]

6. Fungsi ketumpatan tercantum X dan Y diberikan sebagai

$$f(x,y) = \begin{cases} 2 & 0 < x < y < \infty \\ 0 & \text{sebaliknya} \end{cases}$$

(a) Dapatkan fungsi ketumpatan bersyarat X diberikan $Y = y$, $f_{X|Y}(x|y)$.

(b) Hitung $P\left(X < \frac{1}{2} \,\middle|\, Y = \frac{3}{4}\right)$.

(c) Hitung $E[X \mid Y = y]$

(d) Dapatkan $E\{[X - E(X \mid Y = y)]^2 \mid Y = y\}$

[20 markah]

7. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional density function of X given $Y = y$, $f_{X|Y}(x|y)$.

(b) Compute $P\left(X < \frac{1}{2} \,\middle|\, Y = \frac{3}{4}\right)$.

(c) Compute $E[X \mid Y = y]$

(d) Find $E\{[X - E(X \mid Y = y)]^2 \mid Y = y\}$

[20 marks]

7. Andaikan X dan Y adalah dua pembolahubah rawak tak bersandar dengan fungsi ketumpatan kebarangkalian masing-masing

$$f(x) = xe^{-x} \quad \text{bagi } x > 0 \qquad \textit{dan} \qquad f(y) = 2e^{-2y} \quad \text{bagi } y > 0$$

(a) Tuliskan fungsi ketumpatan tercantum X dan Y, $f_{X,Y}(x, y)$

(b) Hitung kebarangkalian $P(X - Y > 2)$

(c) Dapatkan fungsi taburan $F_{X,Y}(x, y)$.

[20 markah]

8. Let X_1 and X_2 be independent exponential random variables, each having parameter λ .

(a) Find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$.

(b) Are Y_1 and Y_2 independent? Justify your answer.

[20 marks]

8. Andaikan X_1 dan X_2 adalah pembolehubah rawak exponen, setiap satu dengan parameter λ .

(a) Dapatkan fungsi ketumpatan tercantum bagi $Y_1 = X_1 + X_2$ dan $Y_2 = \dfrac{X_1}{X_1 + X_2}$.

(b) Adakah Y_1 dan Y_2 tak bersandar? Tentusahkan jawapan anda.

[20 markah]

9. A biased coin is flipped N = 100 times. The probability of getting a head (H) is an unknown value, p. Define

$$X_i = \begin{cases} 1 & if \text{ heads appear} \\ 0 & \text{otherwise} \end{cases}$$

Let $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$.

(a) Show that $E[\overline{X}] = p$

(b) Show that $Var[\overline{X}] = \frac{p(1-p)}{N}$

(c) Approximate the probability that $\overline{X} \in \left[p - \frac{\sqrt{p(1-p)}}{5}, \, p + \frac{\sqrt{p(1-p)}}{5} \right]$

(Hint: Use Central Limit Theorem).

[20 marks]

9. Sekeping syiling tak saksama dilambung sebanyak N = 100 times. Kebarangkalian mendapat suatu kepala (K) ialah p, suatu nilai yang tidak diketahui. Takrifkan

$$X_i = \begin{cases} 1 & \text{jika kepala muncul} \\ 0 & \text{sebaliknya} \end{cases}$$

Andaikan $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$.

(a) *Tunjukkan bahawa* $E[\overline{X}] = p$

(b) *Tunjukkan bahawa* $Var[\overline{X}] = \frac{p(1-p)}{N}$

(c) *Anggarkan kebarangkalian bahawa* $\overline{X} \in \left[p - \frac{\sqrt{p(1-p)}}{5}, \, p + \frac{\sqrt{p(1-p)}}{5} \right]$

(Petunjuk: Guna Teorem Had Memusat).

[20 markah]

10. Let $X = \sum_{i=1}^{N} X_i$, where the X_i's are independent exponential random variables of parameter 1.

(a) Show that the moment generating function of X_1 is $\frac{1}{1-t}$.

(b) Compute the moment generating function of X, $M(t) = E[e^{tX}]$.

(c) Show that for any $a > 0$, $P(X > a) \le e^{-at} M(t)$

(d) Give an upper bound of $P(X > 200)$.

[20 marks]

10. Andaikan $X = \sum_{i=1}^{N} X_i$, di mana X_i adalah pembolehubah rawak exponen dengan parameter 1.

(a) Tunjukkan bahawa fungsi penjana momen bagi X_1 ialah $\frac{1}{1-t}$.

(b) Hitung fungsi penjana momen bagi X, $M(t) = E[e^{tX}]$.

(c) Tunjukkan bahawa bagi sebarang $a > 0$, $P(X > a) \le e^{-at} M(t)$.

(d) Berikan suatu batas atas bagi $P(X > 200)$.

[20 markah]

APPENDIX / *LAMPIRAN*

DISCRETE DISTRIBUTIONS	
Bernoulli	$f(x) = p^x(1-p)^{1-x}, \quad x = 0,1$ $M(t) = 1 - p + pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial	$f(x) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$ $M(t) = (1 - p + pe^t)^n$ $\mu = p, \quad \sigma^2 = np(1-p)$
Geometric	$f(x) = (1-p)^x p, \quad x = 0,1,2,\ldots$ $M(t) = \dfrac{p}{1-(1-p)e^t}, t < \text{In}(1-p)$ $\mu = \dfrac{1-p}{p}, \quad \sigma^2 = \mu = \dfrac{(1-p)}{p^2}$
Negative Binomial	$f(x) = \dfrac{(x+r-p)!}{x!(r-1)!} p^r (1-p)^x, \quad x = 0,1,2,\ldots$ $M(t) = \dfrac{p^r}{\left[1-(1-p)e^t\right]^r}, t < -\text{In}(1-p)$ $\mu = \dfrac{r(1-p)}{p}, \quad \sigma^2 = \dfrac{r(1-p)}{p^2}$
Poisson	$f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$ $M(t) = e^{\lambda(e^t - 1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Hipergeometric	$f(x) = \dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{t}}, \quad x \le r, x \le n_1, r - x \le n_2,$ $\mu = \dfrac{rn_1}{n}, \quad \sigma^2 = \dfrac{rn_1 n_2 (n-r)}{n^2(n-1)}$

CONTINUOUS DISTRIBUTION	
Uniform	$f(x)=\frac{1}{b-a},\quad a\le x\le b$ $M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)},\quad t\ne 0,\quad M(0)=1$ $\mu=\frac{a+b}{2},\quad \sigma^2=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-x/\theta},\ 0\le x\le\infty$ $M(t)=\frac{1}{1-\theta t},\quad t<1/\theta$ $\mu=\theta,\quad \sigma^2=\theta^2$
Gamma	$f(x)=\frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-x/\theta}, 0\le x\le\infty$ $M(t)=\frac{1}{(1-\theta t)^\alpha},\quad t<1/0$ $\mu=\alpha\theta,\quad \sigma^2=\alpha\theta^2$
Chi Square	$f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2},\quad 0\le x\le\infty$ $M(t)=\frac{1}{(1-2t)^{r/2}},\quad t<\frac{1}{2}$ $\mu=r,\quad \sigma^2=2r$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left[(x-\mu)^2/2\sigma^2\right]}, -\infty<x<\infty$ $M(t)=e^{\mu t+\sigma^2t^2/2}$ $E(X)=\mu,\quad Var(X)=\sigma^2$
Beta	$f(x)=\frac{1}{B(\alpha,\beta)}\quad x^{\alpha-1}(1-x)^{\beta-1}, 0<x<1$ $\mu=\frac{\alpha}{\alpha+\beta},\quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

FORMULA	
1.	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
2.	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad \lvert r \rvert < 1$
3.	$\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = (a+b)^n$
4.	$\sum_{x=0}^{n} \binom{n}{x} \binom{r-n}{r-x} = \binom{n}{r}$
5.	$\sum_{x=0}^{n} \binom{n+k-1}{k} w^k = (1-w)^{-n}, \quad \lvert w \rvert < 1$
6.	$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \Gamma(\alpha) = (\alpha-1)!$
7.	$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$
8.	$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
9.	Polar coordinates: $y = r\cos\theta$, $z = r\sin\theta$

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