UNIVERSITI SAINS MALAYSIA

First Semester Examination
2014/2015 Academic Session

December 2014/January 2015

MAT 222 – Differential Equations II
[Persamaan Pembezaan II]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **SIX** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **ENAM** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all five** [5] questions.

*[**Arahan:**   Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Consider the nonlinear system

$$x'(t) = x^2 + 1$$
$$y'(t) = x(y-1)$$

   (i) Find the x-nullcline and the y-nullcline
   (ii) Find the fixed point.

   (b) Let α be a real constant. Given the system

$$x'(t) = y$$
$$y'(t) = ax - 2y$$

   (i) what is the characteristic polynomial of the coefficient matrix for the system
   (ii) compute its eigenvalues and eigenvectors (the answer depends on α, hence the need to break the answer into cases)
   (iii) write down the general solution of the system.

   (c) Explain what is meant when we say the trajectories in the neighbourhood of an equilibrium point is structurally stable.

[100 marks]

1. *(a) Pertimbangkan sistem tak linear berikut*

$$x'(t) = x^2 + 1$$
$$y'(t) = x(y-1)$$

   (i) Carikan nullcline-x dan nullcline-y
   (ii) Dapatkan titik tetap.

   (b) Andaikan α satu pemalar nyata. Diberi sistem

$$x'(t) = y$$
$$y'(t) = ax - 2y$$

   (i) apakah polinom cirian bagi matriks pekali untuk sistem
   (ii) hitungkan nilai eigen dan vektor eigen (jawapan bergantung kepada α, maka perlu untuk bahagikan jawapan kepada kes)
   (iii) tuliskan penyelesaian am bagi sistem.

   (c) Terangkan apa yang dimaksudkan apabila kita mengatakan trajektori dalam kawasan ke jiranan titik keseimbangan stabil secara berstruktur.

[100 markah]

2. (a) Find the solutions of the following system

$$\frac{dx}{dt} = F\left(x, y\right) = x^3 - x$$

$$\frac{dy}{dt} = G\left(x, y\right) = -2y$$

   (i)   Find the critical points
   (ii) Are these points stable?
   (iii) Give a rough sketch of the phase portrait

   (b) In a linearised system, the eigenvalues at a critical point are pure imaginary

   (i)   Describe the trajectories around the critical point
   (ii) If the parameters (constants) in the system are slightly perturbed, what are the effects on the trajectories?

   (c) What is the difference between an autonomous and a non-autonomous system? Give an example of each.

[100 marks]

2. *(a) Cari penyelesaian untuk sistem berikut*

$$\frac{dx}{dt} = F\left(x, y\right) = x^3 - x$$

$$\frac{dy}{dt} = G\left(x, y\right) = -2y$$

   (i)   Carikan titik kritikal
   (ii) Adakah titik-titik ini stabil ?
   (iii) Berikan lakaran kasar potret fasa

   (b) Dalam satu sistem terlinear, nilai eigen pada satu titik kritikal adalah nombor khayalan

   (i)   Perihalkan trajektori-trajektori di sekitar titik kritikal
   (ii) Jika parameter (pemalar) dalam sistem diusik, apakah kesan keatas trajektori?

   (c) Apakah perbezaan di antara sistem otonomi dan sistem bukan otonomi? Berikan contoh bagi setiap satu.

[100 markah]

3. (a) (i) What is an isocline?
      (ii) Given a system

$$\frac{dy}{dx} = -\frac{y}{x}, \qquad x\varepsilon\ [-2,\ 2],\ y\varepsilon\ [-2,2]$$

      sketch a direction field.

   (b) (i) What is a periodic function?
      (ii) Given a periodic function $f(x)$, with period P, it can be represented by a Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

      Find formulae for a_0, a_n and b_n.

   (c) A function $f(x)$ is defined as

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 < x < \pi \end{cases}$$

      (i) Find a Fourier representation of the function
      (ii) Under what theoretical condition wills the given function equals the Fourier representation?

[100 marks]

3. (a) (i) *Apakah yang dimaksudkan dengan secerun?*
      (ii) *Diberi sistem*

$$\frac{dy}{dx} = -\frac{y}{x}, \qquad x\varepsilon\ [-2,\ 2],\ y\varepsilon\ [-2,2]$$

      Lakarkan medan arah.

   (b) (i) *Apakah yang dimaksudkan dengan fungsi berkala?*
      (ii) *Diberi fungsi f berkala (x), dengan kala P, ianya boleh diwakili oleh siri Fourier*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

      Cari formula untuk a_0, a_n dan b_n.

   (c) *Satu fungsi f(x) ditakrifkan sebagai*

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 < x < \pi \end{cases}$$

      (i) *Cari perwakilan Fourier untuk fungsi*
      (ii) *Apakah keadaan teoretis agar fungsi yang diberikan sama dengan perwakilan Fourier?*

[100 markah]

4. (a) The Sturm Liouville operator is defined as

$$L = \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$

Give a description of the Sturm-Liouville eigenvalue problem

(b) Given the following differential equation

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = f(x),$$

put it into Sturm-Liouville form.

(c) Consider the following boundary value problem

$$\frac{d}{dx}\left(x \frac{dy}{dx} \right) + \frac{y}{x} = \frac{1}{x} \quad , \quad x\varepsilon \left[1, \ b \right]$$
$$y(1) \ = \ 0 \ = \ y(b)$$

Write down the solution in terms of the generalised Fourier series expansion.

[100 mark]

4. (a) *Operator Sturm Liouville ditakrifkan sebagai*

$$L = \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$

Berikan satu perihalan mengenai masalah nilai eigen Sturm-Liouville

(b) *Diberi persamaan pembezaan berikut*

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = f(x),$$

letakkannya ke dalam bentuk Sturm-Liouville.

(c) *Pertimbangkan masalah nilai sempadan berikut*

$$\frac{d}{dx}\left(x \frac{dy}{dx} \right) + \frac{y}{x} = \frac{1}{x} \quad , \quad x\varepsilon \left[1, \ b \right]$$
$$y(1) \ = \ 0 \ = \ y(b)$$

Tuliskan penyelesaian dalam sebutan kembangan siri Fourier am.

[100 markah]

5. A partial differential equation problem can be written as

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), \qquad 0 < x < l$$

$$u(x,0) = x - x^2$$

$$\frac{\partial u}{\partial x}(0,t) = 0, u(l,t) = 0$$

    (i)   Evaluate the eigenvalues
    (ii)   Find the corresponding orthonormalised eigenfunctions
    (iii) Find the general solution.

[100 marks]

5. *Satu masalah persamaan pembezaan separa boleh ditulis sebagai.*

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), \qquad 0 < x < l$$

$$u(x,0) = x - x^2$$

$$\frac{\partial u}{\partial x}(0,t) = 0, u(l,t) = 0$$

    (i)   Dapatkan nilai eigen
    (ii)   Dapatkan fungsi eigen pengortonomalan yang sepadan
    (iii) Cari penyelesaian am.

[100 markah]

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