UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2005/2006

Jun 2006

MSS 212E – Further Linear Algebra
[Aljabar Linear Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all six** [6] questions.

***[Arahan** : Jawab **semua enam** [6] soalan.]*

1. (a) Let $A = \begin{pmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & a \\ a & a & \cdots & a & x \end{pmatrix}_{nxn}$.

Find det (A)

[50 marks]

(b) Use determinants to show that for all real values of λ, the only solution of

$$x - 2y = \lambda x$$
$$x - y = \lambda y$$

is $x = 0$ and $y = 0$.

[50 marks]

1. (a) Biar $A = \begin{pmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & a \\ a & a & \cdots & a & x \end{pmatrix}_{nxn}$.

Cari det (A)

[50 markah]

(b) Guna kaedah penentu untuk menunjukkan bagi semua nombor nyata λ, *penyelesaian bagi*

$$x - 2y = \lambda x$$
$$x - y = \lambda y$$

ialah $x = 0$ *dan* $y = 0$ *sahaja.*

[50 markah]

2. Let $F(S, \mathbb{R}) = \{f : S \to \mathbb{R}\}$ where $|S| = 3$.

Let $f_1, f_2 \in F(S, \mathbb{R})$ and $r \in \mathbb{R}$.

Define $(f_1 + f_2)(v) = f_1(v) + f_2(v) \quad \forall v \in S$

and $(r \cdot f_1)(v) = r(f_1(v)) \quad \forall v \in S$

Show that $F(S, \mathbb{R})$ is a vector space over $\mathbb{R}$ with $\dim(F(S, \mathbb{R})) = 3$

[120 marks]

2. *Biar* $F(S,\mathbb{R})=\{f:S\rightarrow\mathbb{R}\}$ *dengan* $|S|=3$.

   Biar $f_1,f_2\in F(S,\mathbb{R})$ *dan* $r\in\mathbb{R}$.

   Takrif $(f_1+f_2)(v)=f_1(v)+f_2(v) \quad \forall v\in S$

   dan $(r\cdot f_1)(v)=r(f_1(v)) \quad \forall v\in S$

   Tunjukkan $F(S,\mathbb{R})$ *ialah suatu ruang vektor atas* $\mathbb{R}$ *dengan* $\dim(F(S,\mathbb{R}))=3$

   [120 markah]

3. Show that $\mathbb{R}^4=\{(a,b,c,d)|a,b,c,d\in\mathbb{R}\}$

   and $M_{2\times2}(\mathbb{R})=\left\{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}|a_1,b_1,c_1,d_1\in\mathbb{R}\right\}$

   are isomorphic by constructing an isomorphism using linear extension method.

   [120 marks]

3. *Tunjukkan* $\mathbb{R}^4=\{(a,b,c,d)|a,b,c,d\in\mathbb{R}\}$

   dan $M_{2\times2}(\mathbb{R})=\left\{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}|a_1,b_1,c_1,d_1\in\mathbb{R}\right\}$

   adalah berisomorfik dengan menggunakan kaedah pengembangan linear

   [120 markah]

4. Let $T:P_2(\mathbb{R})\rightarrow P_2(\mathbb{R})$ such that

   $$(a_0+a_1x+a_2x^2)T=a_0-a_1+(a_1-a_2)x+(a_2-a_0)x^2.$$

   (a) Show that T is a linear transformation.

   [20 marks]

   Let $\alpha=\{1,x,x^2\}$ and $\beta=\{1+x^2,x+x^2,x^2\}$ be two ordered basis of $P_2(\mathbb{R})$

   (b) Find the matrices $T_{\alpha,\alpha}$ and $T_{\beta,\beta}$.

   [50 marks]

   (c) Find $I_{\alpha,\beta}$ and $I_{\beta,\alpha}$ and thus further show that $T_{\alpha,\alpha}$ and $T_{\beta,\beta}$ are similar matrices.

   [60 marks]

4. *Biar* $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ *ditakrif sedemikian hingga*

$$\left(a_0 + a_1x + a_2x^2\right)T = a_0 - a_1 + \left(a_1 - a_2\right)x + \left(a_2 - a_0\right)x^2.$$

(a) *Tunjukkan T ialah suatu transformasi linear.*

[20 markah]

Biar $\alpha = \{1, x, x^2\}$ *dan* $\beta = \{1+x^2, x+x^2, x^2\}$ *merupakan dua asas tertib bagi* $P_2(\mathbb{R})$

(b) *Cari matirks* $T_{\alpha,\alpha}$ *dan* $T_{\beta,\beta}$.

[50 markah]

(c) *Cari* $I_{\alpha,\beta}$ *dan* $I_{\beta,\alpha}$ *dan seterusnya tunjukkan* $T_{\alpha,\alpha}$ *dan* $T_{\beta,\beta}$ *adalah matirks setara.*

[60 markah]

5. Let $T: V \to V$ be a linear transformation and V be a finite dimensional vector space.

(a) Show that $\lambda \in \mathbb{R}$ is a eigen value of T if and only if $T - \lambda 1: V \to V$ is not an isomorphism, where 1 is the identity isomorphism of V.

[30 marks]

(b) If $\lambda \in \mathbb{R}$ is a eigen value of T, then show that the eigen space of T associated to λ, denoted as

$$V_\lambda = \{\text{all eigen vector of } T \text{ associated to } \lambda\} \cup \{\underline{0}\},$$

is a subspace of V, where $\underline{0}$ is the zero vector of V.

[30 marks]

(c) Suppose that $\lambda \in \mathbb{R}$ is an eigen value of T and w_λ is the corresponding eigen vector. Show that w_λ is an eigenvector of T^k corresponding to the eigen value λ^k.

[30 marks]

5. *Biar* $T: V \to V$ *suatu transformasi linear dan* V *adalah suatu ruang vektor berdimensi terhingga.*

(a) *Tunjukkan* $\lambda \in \mathbb{R}$ *adalah suatu nilai eigen bagi* T *jika dan hanya jika*

$$T - \lambda 1: V \to V$$

adalah bukan suatu isomosfisma dengan 1 *adalah suatu isomosfisma identity bagi* V.

[30 markah]

(b) *Jika $\lambda \in \mathbb{R}$ ialah suatu nilai eigen bagi T, tunjukkan ruang vektor eigen bagi T yang sepadan dengan λ iaitu*

$$V_\lambda = \{\textit{semua vektor eigen bagi } T \textit{ yang sepadan dengan } \lambda\} \cup \{\underline{0}\}$$

adalah suatu subruang bagi V, di mana $\underline{0}$ ialah vektor sifar bagi V.

[30 markah]

(c) *Andaikan $\lambda \in \mathbb{R}$ ialah suatu nilai eigen bagi T dan w_λ ialah suatu vektor eigen yang sepadan dengan λ. Tunjukkan w_λ merupakan suatu vektor eigen bagi T^k yang sepadan dengan nilai eigen λ^k.*

[30 markah]

6. Let $T : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ be the linear transformation given by

$$\begin{pmatrix} a \\ b \end{pmatrix} T = \begin{pmatrix} a+2b \\ 2a+b \end{pmatrix}$$

Define $\langle A, B \rangle = \sum_{i=1}^{2} a_{i,1} b_{i,1}$ where $A = (a_{i,j}), B = (b_{i,j}) \in M_{2\times 1}(\mathbb{R})$.

(a) Show that T is self-adjoint

[50 marks]

(b) According to Spectral Theorem, there will exist an orthonomal basis α of $M_{2\times 1}(\mathbb{R})$ such that with respect to α, T is diagonalizable. Find this basis α and show that T is diagonalizable.

[90 marks]

6. *Biar $T : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ merupakan suatu transformasi linear yang di takrif sedemikian hingga*

$$\begin{pmatrix} a \\ b \end{pmatrix} T = \begin{pmatrix} a+2b \\ 2a+b \end{pmatrix}$$

Takrifkan $\langle A, B \rangle = \sum_{i=1}^{2} a_{i,1} b_{i,1}$ di mana $A = (a_{i,j}), B = (b_{i,j}) \in M_{2\times 1}(\mathbb{R})$.

(a) *Tunjukkan T adalah adjoin-sendiri.*

[50 markah]

(b) *Mengikut Teorem Spectral, akan wujud suatu asas ortonormal α bagi $M_{2\times 1}(\mathbb{R})$ sedemikian hingga relative kepada α, T adalah terpepenjuru. Cari asas α ini dan tunjukkan T adalah terpepenjuru.*

[90 markah]

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