UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2013/2014 Academic Session

June 2014

MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) A coin is tossed three times. Let H and T denote head and tail, respectively. Event B is defined as an event where at least two heads are obtained.

   (i) Find the set of all possible outcomes, Ω for event B.

   (ii) From (i), find the σ-field, S of this experiment for event B.

[40 marks]

(b) Assume that C and D are two events defined over the same sample space with probabilities $P(C)=\frac{1}{2}$ and $P(D)=\frac{1}{4}$. Show that $P(C\cup D)\geq\frac{1}{2}$.

[20 marks]

(c) The random variables X and Y are uniformly distributed with the following joint probability density function (pdf):

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{\pi}, \text{ if } x^2+y^2\leq 1\\ 0, \text{ otherwise}\end{cases}.$$

Find the marginal probability density functions of X and Y.

[20 marks]

(d) State at least one property of the cumulative distribution function (cdf) as to why the function $F_X(x)$ below is not a valid cdf.

$$F_X(x)=\begin{cases}0, \ x<1\\ 2x-x^2, \ 1\leq x\leq 2\\ 1, \ x>2\end{cases}$$

[20 marks]

1. (a) Suatu duit syiling dilambungkan tiga kali. Biarkan H dan T mewakili kepala dan bunga, masing-masing. Peristiwa B ditakrifkan sebagai peristiwa sekurang-kurangnya dua kepala diperoleh.

   (i) Cari set bagi semua kesudahan yang mungkin, Ω untuk peristiwa B.

   (ii) Daripada (i), cari medan-σ, S untuk eksperimen ini bagi peristiwa B.

[40 markah]

(b) Andaikan bahawa C dan D adalah dua peristiwa yang ditakrifkan pada ruang sampel yang sama dengan kebarangkalian $P(C)=\frac{1}{2}$ dan $P(D)=\frac{1}{4}$. Tunjukkan bahawa $P(C\cup D)\geq\frac{1}{2}$.

[20 markah]

(c) *Pembolehubah rawak X dan Y adalah bertaburan seragam dengan fungsi ketumpatan kebarangkalian (fkk) tercantum berikut:*

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi}, \text{ jika } x^2 + y^2 \leq 1 \\ 0, \text{ sebaliknya} \end{cases}.$$

Cari fungsi ketumpatan kebarangkalian sut untuk X dan Y.

[20 markah]

(d) *Nyatakan sekurang-kurangnya satu sifat fungsi taburan longgokan (ftl) kenapa fungsi* $F_X(x)$ *di bawah bukan ftl yang sah.*

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 2x - x^2, & 1 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

[20 markah]

2. (a) The moment generating function of the random variable X is

$$M_X(t) = \frac{1}{1 + 2p - 2pe^t},$$

where $e^t < \dfrac{1+2p}{2p}$. Find $E(X)$ and Var(X).

[40 marks]

(b) Let X_1 and X_2 be continuous random variables with the joint pdf $f_{X_1,X_2}(x_1, x_2)$, $-\infty < x_i < \infty$, i = 1, 2. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2$.

(i) Find the joint pdf of Y_1 and Y_2, i.e. $f_{Y_1,Y_2}(y_1, y_2)$.

(ii) Find the marginal pdf of Y_1, i.e. $f_{Y_1}(y_1)$.

[30 marks]

(c) Let $Y_1 < Y_2 < \ldots < Y_n$ represent the order statistics of a random sample of size n from a distribution with pdf $f(x) = \dfrac{3x^2}{\theta^3}$, $0 < x < \theta$, zero elsewhere. Find $P\left(c < \dfrac{Y_n}{\theta} < 1\right)$, where $0 < c < 1$.

[30 marks]

2. (a) *Fungsi penjana momen untuk pembolehubah rawak X ialah*

$$M_X(t) = \frac{1}{1+2p-2pe^t},$$

yang mana $e^t < \dfrac{1+2p}{2p}$. *Cari E(X) dan Var(X).*

[40 markah]

(b) *Biarkan* X_1 *dan* X_2 *sebagai pembolehubah rawak selanjar dengan fkk tercantum* $f_{X_1,X_2}(x_1,x_2)$, $-\infty < x_i < \infty$, i = 1, 2. *Biarkan* $Y_1 = X_1 + X_2$ *dan* $Y_2 = X_2$.

(i) *Cari fkk tercantum bagi* Y_1 *dan* Y_2, *iaitu* $f_{Y_1,Y_2}(y_1,y_2)$.

(ii) *Cari fkk sut bagi* Y_1, *iaitu* $f_{Y_1}(y_1)$.

[30 markah]

(c) *Biarkan* $Y_1 < Y_2 < \ldots < Y_n$ *mewakili statistik tertib untuk sampel rawak saiz* n *daripada taburan dengan fkk* $f(x) = \dfrac{3x^2}{\theta^3}$, $0 < x < \theta$, *sifar di tempat lain.* *Cari* $P\left(c < \dfrac{Y_n}{\theta} < 1\right)$, *yang mana* $0 < c < 1$.

[30 markah]

3. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a $N(\mu, \sigma^2)$ distribution. The sample mean is defined as $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$. Find the distribution of the following statistics:

(i) $\displaystyle\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$.

(ii) $\dfrac{n\left(\overline{X} - \mu\right)}{\sqrt{\sum_{i=1}^{n}(X_i - \mu)^2}}$.

[30 marks]

(b) Let the probability mass function of Y_n be $f_n(y) = 1$, for $y = n$, and zero elsewhere. Show that Y_n does not have a limiting distribution.

[20 marks]

(c) Show that the product of the sample observations is a sufficient statistic for θ (θ > 0) if the random sample is taken from a gamma distribution with parameters α = θ and λ = 6. Note that if $X \sim G(\alpha,\lambda)$, then $f(x) = \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}$, for $x > 0$.

[30 marks]

(d) Let X be a random variable from a binomial, bin(n, θ) distribution. Show that X^2 is not an unbiased estimator of θ.

[20 marks]

3. *(a)* *Biarkan* $X_1, X_2, ..., X_n$ *sebagai pembolehubah rawak tak bersandar dan bertaburan secaman daripada taburan* $N(\mu,\sigma^2)$. *Min sampel ditakrifkan sebagai* $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$. *Cari taburan untuk statistik-statistik berikut:*

(i) $\sum_{i=1}^{n} \left(\dfrac{X_i - \mu}{\sigma} \right)^2$.

(ii) $\dfrac{n(\bar{X} - \mu)}{\sqrt{\sum_{i=1}^{n} (X_i - \mu)^2}}$.

[30 markah]

(b) *Biarkan fungsi jisim kebarangkalian untuk* Y_n *sebagai* $f_n(y) = 1$, *untuk y = n, dan sifar di tempat lain. Tunjukkan bahawa* Y_n *tidak mempunyai taburan penghad.*

[20 markah]

(c) *Tunjukkan bahawa hasil darab cerapan-cerapan sampel adalah statistik cukup untuk* θ (θ > 0) *jika sampel rawak diambil daripada taburan gama dengan parameter* α = θ *dan* λ = 6. *Perhatikan bahawa jika* $X \sim G(\alpha,\lambda)$, *maka* $f(x) = \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}$, *untuk* $x > 0$.

[30 markah]

(d) *Biarkan X sebagai pembolehubah rawak daripada taburan binomial, bin(n,* θ*). Tunjukkan bahawa* X^2 *bukan penganggar saksama untuk* θ.

[20 markah]

4. (a) Twenty motors were put on test under a high temperature setting. The lifetimes in hours of these motors under this temperature setting are as follows:

1	4	5	21	22	28	40	42	51	53
58	67	95	124	124	160	202	260	303	363

Suppose that the lifetime of a motor under this temperature setting has a gamma, $G\left(1, \dfrac{1}{\theta}\right)$ distribution.

(i) Find the maximum likelihood estimator of θ.

(ii) From (i), find the maximum likelihood estimate of θ.

[30 marks]

(b) Assume that $X_1, X_2, ..., X_n$ is a random sample from a normal, $N(\alpha, 1)$ distribution.

(i) Find the Cramer Rao's lower bound for the variance of unbiased estimators of α^2.

(ii) Find the efficiency of the estimator $T = \bar{X}^2 - \dfrac{1}{n}$.

(iii) Is T an asymptotically efficient estimator of α^2? Explain.

[50 marks]

(c) If X has a Poisson distribution with parameter θ, i.e. $X \sim P_o(\theta)$ and the prior distribution of Θ is gamma, $G(\alpha, \lambda)$, α and λ are known, find the posterior distribution of Θ given $X = x$.

[20 marks]

4. (a) Dua puluh buah motor diuji di bawah keadaan suhu yang tinggi. Masa hayat dalam jam untuk motor-motor ini di bawah keadaan suhu ini adalah seperti berikut:

1	4	5	21	22	28	40	42	51	53
58	67	95	124	124	160	202	260	303	363

Andaikan bahawa masa hayat suatu motor di bawah keadaan suhu ini mempunyai taburan gama, $G\left(1, \dfrac{1}{\theta}\right)$.

(i) Cari penganggar kebolehjadian maksimum untuk θ.

(ii) Daripada (i), cari anggaran kebolehjadian maksimum untuk θ.

[30 markah]

(b) Andaikan bahawa $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan normal,* $N(\alpha, 1)$.

(i) Cari batas bawah Rao Cramer untuk varians penganggar-penganggar saksama α^2.

(ii) Cari kecekapan penganggar $T = \bar{X}^2 - \dfrac{1}{n}$.

(iii) *Adakah T suatu penganggar cekap berasimptot untuk* α^2 *? Jelaskan.*

[50 markah]

(c) *Jika X mempunyai taburan Poisson dengan parameter* θ, *iaitu* $X \sim P_o(\theta)$ *dan taburan prior untuk* Θ *ialah gama,* $G(\alpha, \lambda)$, α *dan* λ *adalah diketahui, cari taburan posterior untuk* Θ *diberi* $X = x$.

[20 markah]

5. (a) Assume that a certain random variable has a binomial distribution. If we desire a 90% confidence interval for p that is at most 0.02 in length, find the sample size, n.

(Hint: Note that $\sqrt{\left(\frac{y}{n}\right)\left(1-\frac{y}{n}\right)} \le \sqrt{\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}$)

[20 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = e^{-(x-\theta)},\ x > \theta,\ -\infty < \theta < \infty.$$

The following test is used to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$: Reject H_0 if and only if $Y_1 = \min(X_1, X_2, ..., X_n) > c$. Find the power function of this test.

[20 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from a Poisson, $P_o(\theta)$ distribution having probability mass function (pmf)

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!},\quad x = 0, 1, 2, \dots ;\ \theta > 0.$$

Find the uniformly most powerful test of size $\alpha = 0.1$ for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$. Let $n = 30$ and use the central limit theorem.

[30 marks]

(d) Let X be a single observation from a distribution with pdf

$$f(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1,\ \theta > 0.$$

Find the likelihood ratio test of size-α for testing $H_0 : \theta = 1$ versus $H_1 : \theta \ne 1$.

[30 marks]

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5. (a) *Andaikan bahawa suatu pembolehubah rawak tertentu mempunyai taburan binomial. Jika kita ingini suatu selang keyakinan 90% untuk p dengan panjang sebanyak-banyaknya 0.02, cari saiz sampel, n.*

(Petua: Perhatikan bahawa $\sqrt{\left(\frac{y}{n}\right)\left(1-\frac{y}{n}\right)} \le \sqrt{\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}$*)*

[20 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan dengan fkk*

$$f(x;\theta) = e^{-(x-\theta)},\ x > \theta,\ -\infty < \theta < \infty$$

Ujian berikut digunakan untuk menguji $H_0 : \theta = \theta_0$ *lawan* $H_1 : \theta = \theta_1$ *: Tolak* H_0 *jika dan hanya jika* $Y_1 = \min(X_1, X_2, ..., X_n) > c$*. Cari fungsi kuasa ujian ini.*

[20 markah]

(c) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan Poisson,* $P_o(\theta)$ *yang mempunyai fungsi jisim kebarangkalian (fjk)*

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!},\ \ x = 0, 1, 2, \ldots;\ \theta > 0.$$

Cari ujian paling berkuasa secara seragam saiz $\alpha = 0.1$ *bagi menguji* $H_0 : \theta = 1$ *lawan* $H_1 : \theta > 1$*. Biarkan* $n = 30$ *dan gunakan teorem had memusat.*

[30 markah]

(d) *Biarkan X sebagai cerapan tunggal daripada taburan dengan fkk*

$$f(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1,\ \theta > 0.$$

Cari ujian nisbah kebolehjadian saiz-α *untuk menguji* $H_0 : \theta = 1$ *lawan* $H_1 : \theta \neq 1$.

[30 markah]

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APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x} p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x} x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2} x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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