UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2005/2006

April/May 2006

MSS 212E – Further Linear Algebra
[Aljabar Linear Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SIX pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ENAM muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all nine** [9] questions.

*[**Arahan** : Jawab **semua sembilan** [9] soalan.]*

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1. (a) Let $A = \left(a_{i,j}\right)$ be an $n \times n$ matrix such that

$$a_{i,j} = -a_{j,i}$$

If n is an odd number, show that $\det\left(A\right) = 0$

[35 marks]

(b) Find

$$\det \begin{pmatrix} 1 & x_1 & x_2 & \cdots & \cdots & x_n \\ 1 & x & x_2 & \cdots & \cdots & x_n \\ \vdots & x_1 & x & x_3 & \cdots & x_n \\ & & x_2 & & & \vdots \\ & \vdots & \vdots & & & x_n \\ 1 & x_1 & x_2 & & & x \end{pmatrix}$$

[15 marks]

1. (a) Biar $A = \left(a_{i,j}\right)$ *suatu matriks* $n \times n$ *sedemikian hingga*

$$a_{i,j} = -a_{j,i}$$

Jika n *ialah nombor ganjil, tunjukkan* $\det\left(A\right) = 0$

[35 markah]

(b) Cari

$$\det \begin{pmatrix} 1 & x_1 & x_2 & \cdots & \cdots & x_n \\ 1 & x & x_2 & \cdots & \cdots & x_n \\ \vdots & x_1 & x & x_3 & \cdots & x_n \\ & & x_2 & & & \vdots \\ & \vdots & \vdots & & & x_n \\ 1 & x_1 & x_2 & & & x \end{pmatrix}$$

[15 markah]

2. Let $M_{2\times 1}\left(\mathbb{C}\right) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x = a_1 + b_1 i,\ y = a_2 + b_2 i,\ a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{R} \right\}$

Let $A = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix}$, $B = \begin{pmatrix} c_1 + d_1 i \\ c_2 + d_2 i \end{pmatrix} \in M_{2\times 1}\left(\mathbb{C}\right)$, $r \in \mathbb{R}$.

Define $\quad A + B = \begin{pmatrix} \left(a_1 + c_1\right) + \left(b_1 + d_1\right) i \\ \left(a_2 + c_2\right) + \left(b_2 + d_2\right) i \end{pmatrix}$

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and $\quad rA = \begin{pmatrix} ra_1 + rb_1 i \\ ra_2 + rb_2 i \end{pmatrix}.$

Show that $M_{2\times1}(\mathbb{C})$ is a vector space over $\mathbb{R}$ with $\dim(M_{2\times1}(\mathbb{C})) = 4$.

[120 marks]

2. *Biar* $M_{2\times1}(\mathbb{C}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x = a_1 + b_1 i,\ y = a_2 + b_2 i,\ a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{R} \right\}$

Biar $A = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix},\ B = \begin{pmatrix} c_1 + d_1 i \\ c_2 + d_2 i \end{pmatrix} \in M_{2\times1}(\mathbb{C}),\quad r \in \mathbb{R}.$

Takrifkan $A + B = \begin{pmatrix} (a_1 + c_1) + (b_1 + d_1) i \\ (a_2 + c_2) + (b_2 + d_2) i \end{pmatrix}$

dan $\quad rA = \begin{pmatrix} ra_1 + rb_1 i \\ ra_2 + rb_2 i \end{pmatrix}.$

Tunjukkan $M_{2\times1}(\mathbb{C})$ *ialah suatu ruang vektor atas* $\mathbb{R}$ *sedemikian hingga* $\dim(M_{2\times1}(\mathbb{C})) = 4$.

[120 markah]

3. Let S and T be finite sets such that $|S| = |T|$. Construct an isomorphism to show that $\text{Fun}(S, \mathbb{R})$ is isomorphic to $\text{Fun}(T, \mathbb{R})$.

[100 marks]

3. *Biar* S *dan* T *merupakan set terhingga sedemikian hingga* $|S| = |T|$. *Bentukkan suatu isomorfisma untuk menunjukkan* $\text{Fun}(S, \mathbb{R})$ *adalah berisomorfik dengan* $\text{Fun}(T, \mathbb{R})$.

[100 markah]

4. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation such that

$$(a_0 + a_1 x + a_2 x^2)\, T = a_0 - a_1 + (a_1 - a_2) x + (a_2 - a_0) x^2 .$$

(a) Show that $T_{\alpha,\alpha}$ and $T_{\beta,\beta}$ are similar matrices where $\alpha = \{1, x, x^2\}$ and $\beta = \{1 + x^2, x + x^2, x^2\}$.

[120 marks]

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(b) Find $\det(T)$.

[10 marks]

4. *Biar* $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ *suatu transformasi linear sedemikian hingga*

$$(a_0 + a_1 x + a_2 x^2)T = a_0 - a_1 + (a_1 - a_2)x + (a_2 - a_0)x^2.$$

(a) *Tunjukkan* $T_{\alpha,\alpha}$ *dan* $T_{\beta,\beta}$ *adalah matriks setara jika* $\alpha = \{1, x, x^2\}$ *dan* $\beta = \{1 + x^2, x + x^2, x^2\}$.

[120 markah]

(b) *Cari* $\det(T)$.

[10 markah]

5. Suppose V is a finite dimensional vector space, and $T : V \to V$ is a linear transformation. Let $\lambda \in \mathbb{R}$ is an eigenvalue of T, and suppose E_λ is the set containing all eigenvectors corresponding to λ together with the vector 0.

(a) Check that E_λ is a subspace of V.

[20 marks]

(b) Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigenvalues of T and v_{λ_i} are the corresponding eigenvectors. Show that $v_{\lambda_1}, v_{\lambda_2}, \cdots, v_{\lambda_n}$ are linearly independent.

[40 marks]

(c) Suppose that $S : V \to V$ is a linear transformation satisfying $S \circ T = T \circ S$. Prove $(E_\lambda)S \subseteq E_\lambda$ for every λ.

[20 marks]

5. *Katakan* V *suatu ruang vektor berdimensi terhingga, dan* $T : V \to V$ *suatu transformasi linear. Biar* $\lambda \in \mathbb{R}$ *suatu nilai eigen bagi* T, *dan katakan* E_λ *merupakan set yang mengandungi semua vektor eigen yang sepadan dengan* λ *berserta dengan vektor 0.*

(a) *Tunjukkan bahawa* E_λ *merupakan suatu subruang bagi* V.

[20 markah]

(b) *Katakan* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *merupakan nilai-nilai eigen yang berlainan bagi* T *dan* v_{λ_i} *merupakan vektor-vektor eigen yang sepadan. Tunjukkan bahawa* $v_{\lambda_1}, v_{\lambda_2}, \cdots, v_{\lambda_n}$ *tak bersandar secara linear.*

[40 markah]

.../5-

(c) Katakan $S: V \to V$ suatu transformasi linear sedemikian hingga $S \circ T = T \circ S$. Tunjukkan bahawa $(E_\lambda)S \subseteq E_\lambda$ bagi setiap λ.

[20 markah]

6. Suppose M is a subspace of a finite dimensional inner product space V. Show that

(a) $V = M \oplus M^\perp$ (i.e $V = M + M^\perp$ and $M \cap M^\perp = \{0\}$).

[45 marks]

(b) $(M^\perp)^\perp = M$. [45 marks]

6. Katakan M merupakan suatu subruang bagi suatu ruang hasildarab terkedalam berdimensi terhingga V. Tunjukkan bahawa

(a) $V = M \oplus M^\perp$ (i.e $V = M + M^\perp$ dan $M \cap M^\perp = \{0\}$).

[45 markah]

(b) $(M^\perp)^\perp = M$. *[45 markah]*

7. Investigate the following problem: What happens if the Gram-Schmidt procedure is applied to a list of vectors that are not linearly independent? Prove your answer.

[40 marks]

7. Kaji masalah berikut : Apakah yang akan berlaku jika proses Gram-Schmidt digunakan ke atas suatu senarai vektor-vektor yang bukan tak bersandar secara linear? Buktikan jawapan anda.

[40 markah]

8. Prove that *for every* non-zero $a \in \mathbb{R}$, the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ cannot be diagonalzed.

[30 marks]

8. Buktikan bahawa bagi setiap nilai tak-sifar $a \in \mathbb{R}$, matriks $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ tak terpepenjurukan.

[30 markah]

9. Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$(x_1, x_2, \cdots, x_n)T = (0, x_1, x_2, \cdots, x_{n-1}).$$

(a) Compute the adjoint of T.

(b) Is T seft adjoint? Prove your answer.

[60 marks]

9. Pertimbangkan transformasi linear $T : \mathbb{R}^n \to \mathbb{R}^n$ yang diberi oleh

$$(x_1, x_2, \cdots, x_n)T = (0, x_1, x_2, \cdots, x_{n-1}).$$

(a) Kirakan adjoin bagi T.

(b) Adakah T adjoin-sendiri? Buktikan jawapan anda.

[60 markah]

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