UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2014/2015

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MST 561 - Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **FIVE** (5) questions.

*[**Arahan:** Jawab **LIMA** (5) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Find the moment generating function of the random variable $Y = 3X + 2$, where:

   (i) X is an exponential random variable with parameter λ.

   (ii) X is a binomial random variable with parameters $n = 5$ and $p = \frac{1}{2}$.

[30 marks]

(b) If $E(X) = 1$, $E(Y) = 4$, $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$ and $\rho_{X,Y} = 0.1$, find the covariance of T and U, where $T = 2X + Y$ and $U = 2X - Y$. Here, $\rho_{X,Y}$ is the correlation coefficient of X and Y.

[30 marks]

(c) Let X be a random variable with the following probability density function (pdf):

$$f_X(x) = \frac{1}{2}\alpha e^{-\alpha|x|}, \ -\infty < x < \infty, \ \alpha > 0.$$

Find the pdf of Y in (i) and (ii) using the distribution function method.

   (i) $Y = |X|$.

   (ii) $Y = X^2$.

[40 marks]

1. (a) Cari fungsi penjana momen untuk pembolehubah rawak $Y = 3X + 2$, *yang mana:*

   (i) X ialah pembolehubah rawak eksponen dengan parameter λ.

   (ii) X ialah pembolehubah rawak binomial dengan parameter-parameter n = 5 dan $p = \frac{1}{2}$.

[30 markah]

(b) Jika $E(X) = 1$, $E(Y) = 4$, $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$ dan $\rho_{X,Y} = 0.1$, *cari kovarians T dan U, yang mana* $T = 2X + Y$ *dan* $U = 2X - Y$. *Di sini* $\rho_{X,Y}$ *ialah pekali korelasi X dan Y.*

[30 markah]

(c) *Biarkan X sebagai suatu pembolehubah rawak dengan fungsi ketumpatan kebarangkalian (fkk) berikut:*

$$f_X(x) = \frac{1}{2}\alpha e^{-\alpha|x|}, \ -\infty < x < \infty, \ \alpha > 0.$$

Cari fkk untuk Y dalam (i) dan (ii) dengan menggunakan kaedah fungsi taburan.

(i) $Y = |X|$.

(ii) $Y = X^2$.

[40 markah]

2. (a) Let the joint pdf of X and Y be:

$f(x, y) = x + y, \ 0 < x < 2, 0 < y < 2,$

(i) Find the marginal pdf of X and the marginal pdf of Y.

(ii) Are X and Y stochastically independent? Explain your answer.

[30 marks]

(b) Let X and Y be independent random variables, each having the uniform $U(-1,1)$ distribution. Find the joint pdf of $X + Y$ and $X - Y$.

[30 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample of size n from the uniform $U(0,1)$ distribution and let $Y_1 < Y_2 < ... < Y_n$ represent the corresponding order statistics. If $n = 2k + 1$, $k = 0, 1, 2, \ldots$, then Y_{k+1} is the sample median.

(i) Find the distribution of Y_{k+1}.

(ii) Find the expectation of Y_{k+1}.

(iii) Is the expectation of Y_{k+1} in (ii) equal to the population median? Explain.

[40 marks]

2. (a) *Biarkan fkk tercantum X dan Y sebagai:*

$f(x, y) = x + y, \ 0 < x < 2, 0 < y < 2,$

(i) *Cari fkk sut untuk X dan fkk sut untuk Y.*

(ii) *Adakah X dan Y tak bersandar secara stokastik? Jelaskan jawapan anda.*

[30 markah]

(b) *Biarkan X dan Y sebagai pembolehubah rawak tak bersandar, setiap satu mempunyai taburan seragam* $U(-1,1)$. *Cari fkk tercantum untuk* $X + Y$ *dan* $X - Y$.

[30 markah]

(c) *Andaikan* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak saiz n daripada taburan seragam* $U(0,1)$ *dan biarkan* $Y_1 < Y_2 < ... < Y_n$ *mewakili statistik tertib yang sepadan. Jika* $n = 2k + 1$, $k = 0, 1, 2, \ldots$, *maka* Y_{k+1} *ialah median sampel.*

(i) *Cari taburan* Y_{k+1}.

(ii) *Cari jangkaan* Y_{k+1}.

(iii) *Adakah jangkaan* Y_{k+1} *dalam (ii) bersamaan dengan median populasi? Jelaskan.*

[40 markah]

3. (a) Consider a random sample of size n from the standard normal, $N(0,1)$ distribution. For $k < n$, define:

$$\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i \quad \text{and} \quad \overline{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i \, .$$

Find the distributions of the following statistics:

(i) $\overline{X}_k - \overline{X}_{n-k}$.

(ii) $\overline{X}_k + X_n$.

[40 marks]

(b) Let $Y_1 < Y_2 < ... < Y_n$ represent order statistics for a random sample of size n from a population distribution having the distribution function F and density function f. Find the limiting distribution of $nF(Y_1)$.

[30 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from the uniform $U\left(\mu - \sqrt{2}\sigma, \mu + \sqrt{2}\sigma\right)$ distribution. Find the method of moments estimator of μ and σ.

[30 marks]

3. *(a) Pertimbangkan suatu sampel rawak saiz n daripada taburan normal piawai, N(0,1). Untuk k < n, takrifkan*

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i \text{ dan } \bar{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i .$$

Cari taburan-taburan untuk statistik-statistik berikut:

(i) $\bar{X}_k - \bar{X}_{n-k}$.

(ii) $\bar{X}_k + X_n$.

[40 markah]

(b) Biarkan $Y_1 < Y_2 < ... < Y_n$ *mewakili statistik tertib untuk suatu sampel rawak saiz n daripada suatu taburan populasi yang mempunyai fungsi taburan F dan fungsi ketumpatan f. Cari taburan penghad* $nF(Y_1)$.

[30 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan seragam* $U\left(\mu - \sqrt{2}\sigma, \mu + \sqrt{2}\sigma\right)$. *Cari kaedah penganggar momen untuk* μ *dan* σ.

[30 markah]

4. (a) Assume that $X_1, X_2, ..., X_n$ is a random sample from an exponential distribution with density function:

$$f(x;\theta) = \theta e^{-\theta x}, \, x > 0, \, \theta > 0.$$

(i) Find the maximum likelihood estimator (mle) of $\frac{1}{\theta}$.

(ii) Show that the mle in (i) is an unbiased estimator of $\frac{1}{\theta}$.

(iii) Find an unbiased estimator of $\frac{1}{\theta}$ which is a function of the first order statistic $Y_1 = \min(X_1, X_2, ..., X_n)$.

[40 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from an exponential distribution with density function:

$$f(x;\theta) = \theta e^{-\theta x}, \, x > 0, \, \theta > 0.$$

Find the Cramer Rao's lower bound for the variance of unbiased estimators of $\frac{1}{\theta}$.

[30 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from a distribution with pdf:

$$f(x;\lambda) = \lambda^2 x e^{-\lambda x}, \, x > 0, \, \lambda > 0.$$

(i) Show that $f(x; \lambda)$ is an exponential family of density function.

(ii) By using the information in (i), find a uniformly minimum variance unbiased estimator (UMVUE) of $\frac{1}{\lambda}$.

[30 marks]

4. (a) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan eksponen dengan fungsi ketumpatan:*

$$f\left(x;\theta\right)=\theta e^{-\theta x},\ x>0,\ \theta>0.$$

(i) *Cari penganggar kebolehjadian maksimum (pkm)* $\dfrac{1}{\theta}$.

(ii) *Tunjukkan bahawa pkm dalam (i) ialah suatu penganggar saksama* $\dfrac{1}{\theta}$.

(iii) *Cari penganggar saksama* $\dfrac{1}{\theta}$ *yang merupakan fungsi statistik tertib pertama* $Y_1=\min\left(X_1, X_2, ..., X_n\right)$.

[40 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan eksponen dengan fungsi ketumpatan:*

$$f\left(x;\theta\right)=\theta e^{-\theta x},\ x>0,\ \theta>0.$$

Cari batas bawah Cramer Rao untuk varians penganggar-penganggar saksama $\dfrac{1}{\theta}$.

[30 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak saiz n daripada taburan dengan fkk:*

$$f\left(x;\lambda\right)=\lambda^2 x e^{-\lambda x},\ x>0,\ \lambda>0.$$

(i) *Tunjukkan bahawa* $f(x;\ \lambda)$ *ialah suatu fungsi ketumpatan famili eksponen.*

(ii) *Dengan menggunakan maklumat dalam (i), cari suatu penganggar saksama bervarians minimum secara seragam (PSVMS)* $\dfrac{1}{\lambda}$.

[30 markah]

5. (a) Let $X_1, X_2, ..., X_n$ be a random sample of size n from the uniform $U(\theta, \theta+1)$ distribution. Define $Y_1 = \min(X_1, X_2, ..., X_n)$. Find γ if $(Y_1 - b,\ Y_1 - a)$ is a $100\gamma\%$ confidence interval for θ, where $0 \leq a < b \leq 1$.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pdf:

$$f(x;\theta) = \theta^2 x e^{-\theta x},\ x > 0,\ \theta > 0.$$

Find the uniformly most powerful (UMP) test of size α to test $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from the gamma $G(2, \theta)$ distribution. For testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$, the following test is used: Reject H_0 if and only if $|\bar{X} - 2| \geq c$. Find the value of c so that the test has a size α = 0.05. Assume that n is sufficiently large so that the central limit theorem (CLT) can be used to find an approximate value of c.

[30 marks]

5. *(a)* *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak saiz n daripada taburan seragam* $U(\theta, \theta+1)$. *Takrifkan* $Y_1 = \min(X_1, X_2, ..., X_n)$. *Cari* γ *jika* $(Y_1 - b,\ Y_1 - a)$ *ialah suatu selang keyakinan* $100\gamma\%$ *bagi* θ, *yang mana* $0 \leq a < b \leq 1$.

[30 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan dengan fkk:*

$$f(x;\theta) = \theta^2 x e^{-\theta x},\ x > 0,\ \theta > 0.$$

Cari ujian paling berkuasa secara seragam (UPBS) saiz α *untuk menguji* $H_0 : \theta = 1$ *lawan* $H_1 : \theta > 1$.

[40 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan gama* $G(2, \theta)$. *Untuk menguji* $H_0 : \theta = 1$ *lawan* $H_1 : \theta \neq 1$, *ujian berikut digunakan: Tolak* H_0 *jika dan hanya jika* $|\bar{X} - 2| \geq c$. *Cari nilai c supaya ujian ini mempunyai saiz* α = 0.05. *Andaikan bahawa n adalah cukup besar supaya teorem had memusat (THM) boleh digunakan untuk mencari suatu nilai hampiran c.*

[30 markah]

APPENDIX/ *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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