UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2014/2015 Academic Session

June 2015

MAT 516 – Curve and Surface Methods for CAGD
[Kaedah Lengkung dan Permukaan untuk RGBK]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **THREE** (3) questions.

*[**Arahan**: Jawab **TIGA** (3) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) The algebraic form of a parametric cubic curve **p**, defined on the interval $[0,1]$ can be written as $\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$, where $\mathbf{p}(t)$ is the position vector of any point on the curve and **a**, **b**, **c** and **d** are the vector equivalents of the scalar algebraic coefficients.

(i) Calculate the end-points $\mathbf{p}(0)$ and $\mathbf{p}(1)$, and the corresponding tangent vectors $\mathbf{p}'(0)$ and $\mathbf{p}'(1)$. State your answers in term of **a**, **b**, **c** and **d**.

(ii) Write **a**, **b**, **c** and **d** respectively in term of $\mathbf{p}(0)$, $\mathbf{p}(1)$, $\mathbf{p}'(0)$ and $\mathbf{p}'(1)$. Hence, state the Hermite basis functions $F_1(t),\ F_2(t),\ F_3(t)$ and $F_4(t)$, such that $\mathbf{p}(t) = \mathbf{p}(0)\,F_1(t) + \mathbf{p}'(0)\,F_2(t) + \mathbf{p}'(1)\,F_3(t) + \mathbf{p}(1)F_4(t)$.

(b) Write the Aitken algorithm to construct a polynomial **p** which satisfies the interpolatory constraints $\mathbf{p}(t_i) = \mathbf{p}_i;\ i = 0, \ldots, n$ starting from data points $\mathbf{p}_i = p_i^0$ with corresponding parameter values t_i. Arrange the points $p_i^r;\ r = 1, \ldots, n;\ i = 0, \ldots, n-r$ in a systolic array for the case $n = 3$.

(c) Let a quadratic rational Bézier curve be defined as $r(t) = \dfrac{\sum_{i=0}^{2} w_i p_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}$,

with the control points p_i, weights w_i, and Bernstein polynomials of degree 2, $B_i^2(t) = \dfrac{2!t^i(1-t)^{2-i}}{i!(2-i)!}$, for $i = 0,\ 1,\ 2$.

(i) Let the set of weights be $\{w_0 = 1, w_1, w_2 = 1\}$. Show that when $w_1 = 0$, the curve reduces to the straight line between p_0 and p_1. Subsequently, show that a point $S = r(0.5)$, which is called the shoulder point of the curve, moves along a straight line from $\dfrac{p_0 + p_2}{2}$ to p_1 when w_1 varies from 0 to ∞ (or equivalently when $\dfrac{w_1}{1+w_1}$ varies from 0 to 1).

(ii) Given the three control points $p_0 = (1,0)$, $p_1 = (0,0)$ and $p_2 = (0,1)$, calculate the quadratic rational curve segment defined by these points whose shape is a circular arc spanning 90˚. Sketch the curve together with the control polygon and control points in the same figure.

[100 marks]

1. (a) *Bentuk algebra suatu lengkung kubik berparameter*, $\boldsymbol{p}$, *yang ditakrif pada selang [0,1] boleh ditulis sebagai* $\boldsymbol{p}(t) = \boldsymbol{a}t^3 + \boldsymbol{b}t^2 + \boldsymbol{c}t + \boldsymbol{d}$, *dengan* $\boldsymbol{p}(t)$ *sebagai vektor kedudukan sebarang titik di atas lengkung dan* ***a***, ***b***, ***c*** *dan* ***d*** *adalah kesetaraan vektor bagi pekali algebra skalar.*

(i) *Kira titik hujung,* $\boldsymbol{p}(0)$ *dan* $\boldsymbol{p}(1)$ *dan vektor tangen sepadan* $\boldsymbol{p}'(0)$ *dan* $\boldsymbol{p}'(1)$. *Nyatakan jawapan anda dalam sebutan* ***a***, ***b***, ***c*** *dan* ***d***.

(ii) *Tulis* ***a***, ***b***, ***c*** *dan* ***d*** *masing-masingnya dalam sebutan* $\boldsymbol{p}(0)$, $\boldsymbol{p}(1)$, $\boldsymbol{p}'(0)$ *dan* $\boldsymbol{p}'(1)$. *Seterusnya, kira fungsi asas Hermite* $F_1(t)$, $F_2(t)$, $F_3(t)$ dan $F_4(t)$, supaya

$$\boldsymbol{p}(t) = \boldsymbol{p}(0)\,F_1(t) + \boldsymbol{p}'(0)\,F_2(t) + \boldsymbol{p}'(1)\,F_3(t) + \boldsymbol{p}(1)F_4(t).$$

(b) *Tulis algoritma Aitken bagi membina satu polinomial* $\boldsymbol{p}$ *yang memenuhi kekangan interpolasi* $\boldsymbol{p}(t_i) = \boldsymbol{p}_i;\ i = 0,\dots,n$ *bermula dari titik data* $\boldsymbol{p}_i = p_i^0$ *dengan nilai parameter sepadan* t_i. *Susun titik-titik* $p_i^r;\ r = 1,\dots,n;\ i = 0,\dots,n-r$ *tersebut dalam tatasusunan sistol bagi kes* $n = 3$.

(c) *Andaikan lengkung Bézier nisbah kuadratik ditakrifkan sebagai*

$$r(t) = \frac{\sum_{i=0}^{2} w_i p_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)},$$

dengan titik kawalan p_i, *pemberat* w_i, *dan polinomial Bernstein berdarjah 2,* $B_i^2(t) = \dfrac{2!\,t^i(1-t)^{2-i}}{i!(2-i)!}$, *untuk* $i = 0,\ 1,\ 2.$

(i) *Andaikan set pemberat adalah* $\{w_0 = 1, w_1, w_2 = 1\}$. *Tunjukkan bahawa apabila* $w_1 = 0$, *lengkung terturun kepada garis lurus di antara* p_0 *dan* p_1. *Seterusnya, tunjukkan bahawa titik* $S = r(0.5)$, *yang dipanggil titik bahu lengkung, bergerak sepanjang garis lurus dari* $\frac{p_0 + p_2}{2}$ *ke* p_1 *apabila* w_1 *berubah dari 0 ke* ∞ *(atau secara setara apabila* $\frac{w_1}{1+w_1}$ *berubah dari* 0 *ke* 1*).*

(ii) *Diberi 3 titik kawalan* $p_0 = (1,0)$, $p_1 = (0,0)$ *dan* $p_2 = (0,1)$, *kira tembereng lengkung nisbah kuadratik yang bentuknya adalah lengkok bulatan yang merentang* 90° *yang ditakrifkan oleh titik-titik tersebut. Lakar lengkung bersama-sama poligon dan titik kawalan pada rajah yang sama.*

[100 markah]

2. (a) A general cubic uniform B-spline in segment *i* can be written as

$$P_i(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \\ V_{i+2} \end{bmatrix}$$

where $u \in [0,1]$, and V_i are de Boor control points.

(i) Determine the start and end points of segment *i*.

(ii) Find the tangent vectors at both ends of segment *i*.

(iii) Find the second derivative of a general cubic uniform B-spline in segment *i*. Then, show that two adjacent cubic uniform B-splines of segments *i* and (*i* + 1) are connected with C^2 continuity.

(b) The normalized B-spline basis functions, $N_i^k(t)$ of order *k* are defined recursively by

$$N_i^1(t) = \begin{cases} 1, \text{ if } t \in [t_i, t_{i+1}) \\ 0, \text{ otherwise} \end{cases}$$

and $N_i^k(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_i^{k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t)$, where $0 \le i \le n$ and $T = \{t_0, t_1, \cdots, t_{n+k}\}$ is a knot vector.

(i) Calculate and plot the basis functions, $N_i^3(t), i = 0, 1, 2, 3$ when $T = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(ii) Show that when $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$, then $N_i^3(t) = B_i^3(t), i = 0, 1, 2, 3$ for $t \in [0,1]$, where $B_i^3(t) = \dfrac{3! t^i (1-t)^{3-i}}{(3-i)! i!}$ are Bernstein polynomials of degree 3.

(c) The bicubic B-spline patch is defined by 16 control points, $V_{ij}, i = 0,1,2,3; j = 0,1,2,3$ and is given by $P(u,v) = (\frac{1}{6})^2 UMCM^T V^T$ where

$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$, $V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$ and

$$C = \begin{bmatrix} V_{00} & V_{01} & V_{02} & V_{03} \\ V_{10} & V_{11} & V_{12} & V_{13} \\ V_{20} & V_{21} & V_{22} & V_{23} \\ V_{30} & V_{31} & V_{32} & V_{33} \end{bmatrix}.$$

Write its four corners $P(0,0), P(0,1), P(1,0)$ and $P(1,1)$ as a barycentric sum of nine control points.

[100 marks]

2. (a) *Suatu splin-B seragam kubik am dalam segmen i boleh ditulis sebagai*

$$P_i(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \\ V_{i+2} \end{bmatrix}$$

dengan $u \in [0,1]$, *dan* V_i *adalah titik-titik kawalan de Boor.*

(i) Tentukan titik mula dan titik akhir segmen i.

(ii) Cari vektor tangen pada kedua-dua hujung segmen i.

(iii) *Cari terbitan kedua suatu splin-B seragam kubik am dalam segmen i. Seterusnya, tunjukkan bahawa dua splin-B seragam kubik bersebelahan pada segmen i dan (i + 1) bersambung dengan keselanjaran* C^2.

(b) *Fungsi asas splin-B ternormal,* $N_i^k(t)$ *peringkat k ditakrif secara rekursi oleh*

$$N_i^1(t) = \begin{cases} 1, \text{ jika } t \in [t_i, t_{i+1}) \\ 0, \text{selainnya} \end{cases}$$

dan $N_i^k(t) = \dfrac{t - t_i}{t_{i+k-1}} N_i^{k-1}(t) + \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t)$ *dengan* $0 \le i \le n$ *dan* $T = \{t_0, t_1, \cdots, t_{n+k}\}$ *adalah vektor knot (simpulan).*

(i) *Kira dan lakar fungsi-fungsi asas,* $N_i^3(t), i = 0, 1, 2, 3$ *apabila* $T = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(ii) *Tunjukkan bahawa apabila* $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$, *maka* $N_i^3(t) = B_i^3(t), i = 0, 1, 2, 3$ *untuk* $t \in [0,1]$, *dengan* $B_i^3(t) = \dfrac{3!t^i(1-t)^{3-i}}{(3-i)!i!}$ *adalah polinomial Bernstein darjah 3.*

(c) *Tampalan splin-B bikubik ditakrifkan oleh 16 titik kawalan,* $V_{ij}, i = 0,1,2,3; j = 0,1,2,3$ *dan diberikan sebagai* $P(u,v) = (\frac{1}{6})^2 UMCM^T V^T$

dengan $U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$, $V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$,

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \text{ dan } C = \begin{bmatrix} V_{00} & V_{01} & V_{02} & V_{03} \\ V_{10} & V_{11} & V_{12} & V_{13} \\ V_{20} & V_{21} & V_{22} & V_{23} \\ V_{30} & V_{31} & V_{32} & V_{33} \end{bmatrix}.$$

Tulis empat bucunya $P(0,0), P(0,1), P(1,0)$ *dan* $P(1,1)$ *sebagai hasil tambah baripusat sembilan titik kawalan.*

[100 markah]

3. (a) A Bézier patch of degree $m \times n$ is defined by $\sum_{i=0}^{m}\sum_{j=0}^{n} V_{ij} B_i^m(u) B_j^n(v)$ for $i = 0,\ldots,m;\ j = 0,\ldots,n$ where $0 \le u, v \le 1$, $B_k^s(t) = \dfrac{s!t^k(1-t)^{s-k}}{k!(s-k)!}$ are Bernstein polynomials of degree s and V_{ij} are its Bézier control points.

(i) State the entries of matrices M and W respectively, when the biquadratic Bézier patch $(m = n = 2)$ is written as the matrix-product $UMWM^TV^T$, where $U = \begin{bmatrix} 1 & u & u^2 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & v & v^2 \end{bmatrix}$.

(ii) When the degree of Bézier patch is increased to $(m+1)$ in the u direction, i.e. $\sum_{i=0}^{m+1}\sum_{j=0}^{n} \hat{V}_{ij} B_i^{m+1}(u) B_j^n(v) = \sum_{i=0}^{m}\sum_{j=0}^{n} V_{ij} B_i^m(u) B_j^n(v)$,

show that $\hat{V}_{ij} = \dfrac{i}{m+1} V_{(i-1)j} + (1 - \dfrac{i}{m+1}) V_{ij},\ i = 0,..,m+1;\ j = 0,..,n.$

Then, use this result to state the relation between the new control points and V_{ij} when the degree is raised to $(n+1)$ in the v direction.

(b) Given the four corner points

$P(0,0) = (-1,-1,0), P(0,1) = (-1,1,0), P(1,0) = (1,-1,0)$ and $P(1,1) = (1,1,1)$, of a linear Coons surface $P(u,v)$, calculate the four boundary curves:

(i) $P(0,v)$: the straight line from $P(0,0) = (-1,-1,0)$ to $P(0,1) = (-1,1,0)$;

(ii) $P(u,0)$: the uniform quadratic Lagrange polynomial with $\left\{t_0 = 0, t_1 = \dfrac{1}{2}, t_2 = 1\right\}$ determined by the interpolating points $P_0 = P(0,0) = (-1,-1,0)$, $P_1 = (0,-1,-0.5)$ and $P_2 = P(1,0) = (1,-1,0)$;

(iii) $P(u,1)$: the quadratic Bézier polynomial determined by the control points
$P_0 = P(0,1) = (-1,1,0)$, $P_1 = P(1,1) = (1,1,1)$ and $P_2 = (0,1,0.5)$;

(iv) $P(1,v)$: the cubic Bézier polynomial determined by the control points
$P_0 = P(1,0) = (1,-1,0)$, $P_1 = P(1,1) = (1,1,1)$, $P_2 = (1, \ -0.5, \ 0.5)$ and $P_3 = (1, 0.5, -0.5)$.

Then, discuss the interpolation method to generate a linear Coons surface $P(u,v)$ with these four boundary curves, for $0 \le u, v \le 1$.

[100 marks]

3. (a) Suatu tampalan Bézier berdarjah $m \times n$ ditakrifkan oleh $\sum_{i=0}^{m}\sum_{j=0}^{n} V_{ij} B_i^m(u) B_j^n(v)$ untuk $i = 0,\ldots,m;\ j = 0,\ldots,n$ dengan $0 \le u, v \le 1,\ B_k^s(t) = \dfrac{s!t^k(1-t)^{s-k}}{k!(s-k)!}$ adalah polinomial Bernstein berdarjah s dan V_{ij} adalah titik kawalan Bézier.

(i) Nyatakan pemasukkan matriks M dan W masing-masingnya, apabila tampalan Bézier bikuadratik $(m = n = 2)$ ditulis sebagai hasil darab matriks $UMWM^TV^T$, dengan $U = \begin{bmatrix} 1 & u & u^2 \end{bmatrix}$ dan $V = \begin{bmatrix} 1 & v & v^2 \end{bmatrix}$.

(ii) Apabila darjah tampalan Bézier ditingkatkan kepada $(m+1)$ dalam arah u, iaitu, $\sum_{i=0}^{m+1}\sum_{j=0}^{n} \hat{V}_{ij}\, B_i^{m+1}(u) B_j^n(v) = \sum_{i=0}^{m}\sum_{j=0}^{n} V_{ij} B_i^m(u) B_j^n(v),$ tunjukkan bahawa $\hat{V}_{ij} = \dfrac{i}{m+1} V_{(i-1)j} + (1 - \dfrac{i}{m+1}) V_{ij},\ i = 0,..,m+1;\ j = 0,..,n.$

Seterusnya, guna keputusan ini untuk menyatakan hubungan di antara titik kawalan baru dan V_{ij} apabila darjah ditingkatkan kepada $(n+1)$ dalam arah v.

(b) *Diberi empat titik penjuru*

$P(0,0)=(-1,-1,0), P(0,1)=(-1,1,0), P(1,0)=(1,-1,0)$ *dan*
$P(1,1)=(1,1,1)$, *bagi suatu permukaan Coons linear* $P(u,v)$, *kira empat lengkung sempadan:*

(i) $P(0,v)$: *garis lurus dari titik* $P(0,0)=(-1,-1,0)$ *ke titik* $P(0,1)=(-1,1,0)$;

(ii) $P(u,0)$: *polinomial Lagrange kuadratik seragam dengan* $\left\{t_0=0, t_1=\frac{1}{2}, t_2=1\right\}$ *yang ditentukan oleh titik-titik interpolasi* $P_0=P(0,0)=(-1,-1,0)$, $P_1=(0,-1,-0.5)$ *dan* $P_2=P(1,0)=(1,-1,0)$;

(iii) $P(u,1)$: *polinomial Bézier kuadratik yang ditentukan oleh titik-titik kawalan* $P_0=P(0,1)=(-1,1,0)$, $P_1=P(1,1)=(1,1,1)$ *dan* $P_2=(0,1,0.5)$;

(iv) $P(1,v)$: *polinomial Bézier kubik yang ditentukan oleh titik-titik kawalan* $P_0=P(1,0)=(1,-1,0)$, $P_1=P(1,1)=(1,1,1)$, $P_2=(1, -0.5, 0.5)$ *dan* $P_3=(1,0.5,-0.5)$.

Seterusnya, bincang kaedah interpolasi untuk menjana permukaan Coons linear $P(u,v)$ *dengan empat lengkung sempadan tersebut, untuk* $0 \le u, v \le 1$.

[100 markah]

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