UNIVERSITI SAINS MALAYSIA

First Semester Examination
2013/2014 Academic Session

December 2013 / January 2014

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Consider the linear advection equation

$$C_t + uC_x = 0$$

where C is the dependent variable, x and t are the independent variables. u is a constant.

(a) By considering $C_t = -uC_x$ and taking partial derivatives with respect to time on both sides, show that $C_{tt} = u^2 C_{xx}$

(b) By considering the Taylor expansion of $C(x_i, t_n + \Delta t)$ and making use of $C_{tt} = u^2 C_{xx}$, show that $C_t = \dfrac{C_i^{n+1} - C_i^n}{\Delta t} - 0.5\Delta t u^2 C_{xx}$

(c) Construct a scheme for the linear advection equation using the above expression for C_t and central difference for all spatial derivatives.

[100 marks]

1. Pertimbangkan persamaan adveksi linear

$$C_t + uC_x = 0$$

dengan C pembolehubah bersandar, x dan t pembolehubah tak bersandar. u ialah pemalar.

(a) Dengan mempertimbangkan $C_t = -uC_x$ *dan mengambil terbitan separa terhadap masa pada kedua-dua belah, tunjukkan* $C_{tt} = u^2 C_{xx}$

(b) Dengan mempertimbangkan perkembangan Taylor bagi $C(x_i, t_n + \Delta t)$ *dan mengunakan* $C_{tt} = u^2 C_{xx}$, *tunjukkan* $C_t = \dfrac{C_i^{n+1} - C_i^n}{\Delta t} - 0.5\Delta t u^2 C_{xx}$

(c) Bangunkan suatu skema untuk persamaan adveksi linear dengan menggunakan ungkapan di atas untuk C_t *dan beza pusat untuk semua terbitan ruang.*

[100 markah]

2. Consider the equation

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \qquad \sigma > 0 \text{ is a positive constant.}$$

The DuFort-Frankel scheme for the above equation replaces $2u_i^n$ in the central difference approximation of $\frac{\partial^2 u}{\partial x^2}$ by $u_i^{n+1} + u_i^{n-1}$ and also uses the central difference approximation for $\frac{\partial u}{\partial t}$.

(a) Write the DuFort -Frankel scheme for $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$

(b) Calculate the truncation error of the scheme

[100 marks]

2. Pertimbangkan persamaan

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \qquad \sigma > 0 \textit{ ialah pemalar positif.}$$

Skema DuFort-Frankel untuk persamaan di atas menggantikan $2u_i^n$ *dalam hampiran beza pusat* $\frac{\partial^2 u}{\partial x^2}$ *dengan* $u_i^{n+1} + u_i^{n-1}$ *dan juga menggunakan beza pusat untuk* $\frac{\partial u}{\partial t}$.

(a) Tulis skema DuFort-Frankel untuk $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$

(b) Kira ralat pangkasan skema

[100 markah]

3. (a) Consider the problem of torsion. The torsion function ϕ satisfies the Poisson's equation $\nabla^2\phi + 2 = 0$ with $\phi = 0$ on the boundary. Find ϕ over the cross-section of a rectangular bar bounded by $x = 0, y = 0, x = 3, y = 4$. Taking the mesh size of 1 unit in both the directions, solve the Poisson's equations, using Successive Over Relaxation (S.O.R) method until 2 iterations.

(i) Find $\rho(L_{\omega_b})$ for this system

(ii) Calculate the rate of the convergence $R_\infty(L_{\omega_b})$

(iii) Estimate the optimal relaxation factor ω_b for S.O.R and

(iv) Compare the number of iterations between S.O.R method and Gauss Seidel method for this system.($\varepsilon = 10^{-7}$)

(b) Given B as an $m \times m$ matrix with m distinct eigenvalues; $\alpha_1, \alpha_2, \ldots, \alpha_m$ and corresponding eigenvectors $\beta_1, \beta_2, \ldots, \beta_m$. Suppose A is an $n \times n$ block tridiagonal matrix of the form

$$A = \begin{pmatrix} B & I & & & & \\ I & B & I & & & \\ & I & B & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & B & I \\ & & & & I & B \end{pmatrix}.$$

Find the eigenvalues of A and the corresponding eigenvectors.

[100 marks]

3. (a) Pertimbangkan masalah kilasan berikut. Fungsi kilasan ϕ memenuhi persamaan Poisson iaitu $\nabla^2\phi + 2 = 0$ dimana $\phi = 0$ pada sempadan. Dapatkan ϕ pada keratan rentas satu batang segiempat tepat disempadani oleh $x = 0, y = 0, x = 3, y = 4$. Dengan mengambil saiz mesh 1 unit pada kedua-dua arah, selesaikan persamaan Poisson ini menggunakan kaedah Pengenduran Berlebihan Berturut-turut (P.B.B) sehingga 2 lelaran.

(i) Dapatkan $\rho(L_{\omega_b})$ bagi sistem ini

(ii) Kira kadar penumpuan $R_\infty(L_{\omega_b})$ bagi kaedah ini

(iii) Anggar faktor pengenduran optimal ω_b untuk kaedah P.B.B ini dan

(iv) Bandingkan bilangan lelaran antara kaedah P.B.B dan kaedah Gauss Seidel bagi sistem ini ($\varepsilon = 10^{-7}$).

(b) *Diberi* B *sebagai matriks* $m \times m$ *dengan* m *nilai eigen yang berbeza;* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *dan vektor eigen yang sepadan* $\beta_1, \beta_2, \ldots, \beta_m$. *Andaikan* A *dalah* $n \times n$ *matriks blok tiga pepenjuru dalam bentuk*

$$A = \begin{pmatrix} B & I & & & & \\ I & B & I & & & \\ & I & B & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & B & I \\ & & & & I & B \end{pmatrix}.$$

Dapatkan nilai eigen A *dan vektor eigen yang sepadan.*

[100 markah]

4. (a) Calculate the eigenvalues of the S.O.R iteration matrix corresponding to matrix $A = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}$

(b) Given the block system $\sum_{j=1}^{4} A_{ij} \vec{U}_j = \vec{C}_i; i = 1, 2, 3, 4.$ Write the block Jacobi iterative method and the corresponding block S.O.R iterative method in solving this system.

(c) Show that the weighted Jacobi method defined by $D\vec{x}^{(n+1)} = (1-\omega)D\vec{x}^{(n)} + \omega(\vec{b} + (L+U)\vec{x}^{(n)})$ has the iteration matrix $G = [(1-\omega)I + \omega D^{-1}(L+U)]$.

[100 marks]

4. *(a)* *Kirakan nilai eigen bagi lelaran P.B.B untuk matriks* $A = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}$ *berikut.*

(b) *Diberikan sistem blok* $\sum_{j=1}^{4} A_{ij} \vec{U}_j = \vec{C}_i; i = 1, 2, 3, 4.$ *Tuliskan kaedah lelaran blok Jacobi dan kaedah lelaran blok P.B.B yang sepadan dalam menyelesaikan masalah ini.*

(c) *Tunjukkan kaedah Jacobi berpemberat yang didefinisikan oleh* $D\vec{x}^{(n+1)} = (1-\omega)D\vec{x}^{(n)} + \omega(\vec{b} + (L+U)\vec{x}^{(n)})$ *mempunyai matriks lelaran* $G = [(1-\omega)I + \omega D^{-1}(L+U)]$.

[100 markah]

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