UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2014/2015 Academic Session

June 2015

MGM 501 Real Analysis
[Analisis Nyata]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[Arahan: Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]

1. (a) Let a, b and c be real numbers, with $c \neq 0$. Show that if $a \cdot c = b \cdot c$, then $a = b$.
   (b) Let E be a subset of $\mathbb{R}$. If x is an element of E and is also an upper bound of E, show that x is a supremum of E.
   (c) Why does the set $E = \{2 - (-1)^n/n^2 : n \in \mathbb{N}\}$ have a supremum? Justify your answer. Then determine the supremum of E.

[20 marks]

1. (a) Andaikan a, b dan c nombor nyata, dengan $c \neq 0$. Tunjukkan bahawa jika $a \cdot c = b \cdot c$, maka $a = b$.
   (b) Andaikan E suatu subset kepada $\mathbb{R}$. Jika x ialah unsur dalam E dan juga batas atas E, tunjukkan bahawa x ialah supremum E.
   (c) Mengapa set $E = \{2-(-1)^n/n^2 : n \in \mathbb{N}\}$ mempunyai supremum? Jelaskan jawapan anda. Kemudian tentukan supremum E.

[20 markah]

2. (a) Suppose that $f : [0,1] \to [0,1]$ is continuous. Show that there exists c in $[0,1]$ such that $f(c) = c$.
   (b) Suppose that f and g are continuous functions on a closed interval $[a,b]$, with $g(x) \geq 0$ for all x in $[a,b]$. Show that there exists c in $[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

[20 marks]

2. *(a) Andaikan $f : [0,1] \to [0,1]$ selanjar. Tunjukkan bahawa wujud c dalam $[0,1]$ sedemikian $f(c) = c$.*
   (b) Andaikan f dan g fungsi selanjar pada selang tertutup $[a,b]$, dengan $g(x) \geq 0$ untuk semua x dalam $[a,b]$. Tunjukkan bahawa wujud c dalam $[a,b]$ sedemikian

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

[20 markah]

3. (a) Consider the sequence defined by

$$a_1 = 3, \quad a_{n+1} = 3 - \frac{2}{a_n}, \ n \geq 1.$$

(i) Show, by induction, $a_n \geq 2$ for all $n \in \mathbb{N}$.
(ii) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ decreases.
(iii) Does $\lim_{n\to\infty} a_n$ exist? Give your reason. If the limit exists, find that limit.

(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n = \frac{(n-1)\cos(n^2+n+1)}{2n-1}$$

is bounded by 1. Does $\{a_n\}_{n=1}^{\infty}$ have a convergent subsequence? Give your reason.

[20 marks]

3. *(a) Pertimbangkan jujukan yang ditakrif oleh*

$$a_1 = 3, \quad a_{n+1} = 3 - \frac{2}{a_n}, \ n \geq 1.$$

(i) Tunjukkan, secara aruhan, $a_n \geq 2$ untuk semua $n \in \mathbb{N}$.
(ii) Tunjukkan jujukan $\{a_n\}_{n=1}^{\infty}$ menyusut.
(iii) Adakah $\lim_{n\to\infty} a_n$ wujud? Berikan alasan anda. Jika had itu wujud, dapatkan had tersebut.

(b) Tunjukkan bahawa jujukan $\{a_n\}_{n=1}^{\infty}$ yang diberi oleh

$$a_n = \frac{(n-1)\cos(n^2+n+1)}{2n-1}$$

dibatasi oleh 1. Adakah $\{a_n\}_{n=1}^{\infty}$ mempunyai subjujukan menumpu? Berikan alasan anda.

[20 markah]

4. (a) For any real number a, show that there exist a sequence $\{r_n\}$ of rationals and a sequence $\{i_n\}$ of irrationals such that each of them converges to a.
   (b) Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0, & \text{irrational } x; \\ 1, & \text{rational } x. \end{cases}$$

   Show that f does not have a limit at any point a in $(0,1)$.
   (c) Let $g : (0,1] \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} 0, & \text{irrational } x; \\ 1/n, & \text{rational } x = m/n. \end{cases}$$

   Show that f is continuous only on the irrationals in $(0,1)$.

[20 marks]

4. (a) Untuk setiap nombor nyata a, tunjukkan bahawa wujud jujukan nombor nisbah $\{r_n\}$ dan jujukan nombor bukan nisbah $\{i_n\}$ masing-masing menumpu kepada a.
(b) Andaikan $f : [0,1] \to \mathbb{R}$ fungsi yang ditakrif sebagai

$$f(x) = \begin{cases} 0, & x \text{ bukan nombor nisbah}; \\ 1, & x \text{ nombor nisbah}. \end{cases}$$

Tunjukkan bahawa f tidak mempunyai had pada sebarang titik a dalam $(0,1)$.
(c) Andaikan $g : (0,1] \to \mathbb{R}$ fungsi yang ditakrif sebagai

$$g(x) = \begin{cases} 0, & x \text{ bukan nombor nisbah}; \\ 1/n, & x = m/n \text{ nombor nisbah}. \end{cases}$$

Tunjukkan bahawa f hanya selanjar pada nombor bukan nisbah dalam $(0,1)$.

[20 markah]

5. (a) Prove that if f is differentiable at a point a, then f is continuous at a. Give an example of a function which is continuous at a but is not differentiable at a.

   (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Find F' if

$$F(x) = \int_{x^3}^{2} f(t)\, dt.$$

   (c) Suppose that f is nonnegative and continuous on $[1, 2]$, and that $\int_1^2 x^k f(x)\, dx = 5 + k^2$ for $k = 0, 1, 2$. Show that

$$\int_{\frac{1}{\sqrt{2}}}^{1} x^2 f\left(\frac{1}{x^2}\right)\, dx \leq \frac{5}{2}.$$

   Also find the value of

$$\int_0^1 x^2 f(x+1)\, dx.$$

[20 marks]

5. *(a) Buktikan bahawa jika f terbezakan pada suatu titik a, maka f selanjar pada a. Beri satu contoh fungsi yang selanjar pada a tetapi tak terbezakan pada a.*

   (b) Andaikan $f : \mathbb{R} \to \mathbb{R}$ selanjar. Dapatkan F' jika

$$F(x) = \int_{x^3}^{2} f(t)\, dt.$$

   (c) Andaikan f tak negatif dan selanjar pada $[1, 2]$, dan $\int_1^2 x^k f(x)\, dx = 5 + k^2$ untuk $k = 0, 1, 2$. Tunjukkan bahawa

$$\int_{\frac{1}{\sqrt{2}}}^{1} x^2 f\left(\frac{1}{x^2}\right)\, dx \leq \frac{5}{2}.$$

   Juga, cari nilai untuk

$$\int_0^1 x^2 f(x+1)\, dx.$$

[20 markah]

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