UNIVARIATE AND MULTIVARIATE SYNTHETIC CONTROL CHARTS FOR MONITORING THE PROCESS MEAN OF SKEWED DISTRIBUTIONS

by

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CARTA–CARTA KAWALAN SINTETIK UNIVARIAT DAN MULTIVARIAT UNTUK KAWALAN MIN PROSES TABURAN–TABURAN PENCONGAN

ABSTRAK

Alat yang paling berkuasa dalam Kawalan Kualiti Berstatistik *(SQC)* ialah carta kawalan. Kini, carta-carta kawalan luas diterima dan digunakan di industri. Salah satu penambahbaikan terkini carta-carta $\bar{X}$ Shewhart univariat dan T^2 multivariat adalah perluasan carta-carta sedemikian kepada carta sintetik setara masing-masing dengan menggabungkan setiap daripada carta tersebut dengan carta *conforming run length (CRL)*. Carta-carta sintetik $\bar{X}$ univariat dan T^2 multivariat mengandaikan bahawa proses pendasar mempunyai taburan normal. Walau bagaimanapun, dalam kebanyakan situasi sebenar, andaian kenormalan mungkin tidak dapat dipatuhi. Tesis ini mencadangkan dua carta kawalan sintetik baru untuk populasi pencongan, iaitu carta-carta $\mathrm{WV}-\bar{X}$ sintetik univariat dan $\mathrm{WSD}-T^2$ sintetik multivariat. Carta $\mathrm{WV}-\bar{X}$ sintetik univariat adalah berdasarkan kaedah varians berpemberat manakala carta $\mathrm{WSD}-T^2$ sintetik multivariat menggunakan pendekatan sisihan piawai berpemberat. Kedua-dua carta sintetik baru yang dicadangkan ini berubah menjadi carta-carta $\bar{X}$ sintetik dan T^2 sintetik multivariat apabila taburan pendasar adalah masing-masing univariat dan multivariat normal. Untuk membandingkan prestasi kedua-dua carta baru yang dicadangkan dengan semua carta yang sedia ada bagi taburan pencongan, kadar isyarat palsu dan kadar pengesanan anjakan dalam min dikira. Pada keseluruhannya, keputusan simulasi menunjukkan bahawa carta $\mathrm{WV}-\bar{X}$ sintetik univariat dan carta $\mathrm{WSD}-T^2$ sintetik multivariat mempunyai prestasi yang lebih baik daripada carta-carta setara yang lain dalam literatur.

UNIVARIATE AND MULTIVARIATE SYNTHETIC CONTROL CHARTS FOR MONITORING THE PROCESS MEAN OF SKEWED DISTRIBUTIONS

ABSTRACT

The most powerful tool in Statistical Quality Control (SQC) is the control chart. Control charts are now widely accepted and used in industries. One of the recent enhancements on the univariate Shewhart $\bar{X}$ and multivariate T^2 charts is the extension of these charts to their respective synthetic chart counterparts by combining each of these charts with the conforming run length (CRL) chart. These univariate $\bar{X}$ and multivariate T^2 synthetic charts assume that the underlying process follows a normal distribution. However, in many real situations the normality assumption may not hold. This thesis proposes two new synthetic control charts for skewed populations, which are the univariate synthetic $\mathrm{WV}-\bar{X}$ and the multivariate synthetic $\mathrm{WSD}-T^2$ charts. The univariate synthetic $\mathrm{WV}-\bar{X}$ chart is based on the weighted variance method while the multivariate synthetic $\mathrm{WSD}-T^2$ chart employs the weighted standard deviation approach. These two new proposed synthetic charts reduce to the univariate $\bar{X}$ and multivariate T^2 synthetic charts, when the underlying distributions are univariate and multivariate normal, respectively. To compare the performances of the two new proposed charts with all the existing charts for skewed distributions, the false alarm and mean shift detection rates are computed. Overall, the simulation results show that the proposed univariate synthetic $\mathrm{WV}-\bar{X}$ chart and multivariate synthetic $\mathrm{WSD}-T^2$ chart outperform their respective counterparts found in the literature.

CHAPTER 1
INTRODUCTION

1.1 Statistical Process Control

Statistical process control (SPC) involves the use of a collection of problem solving tools to achieve process stability and improving capability by reducing variability (Montgomery, 2005).

Process improvements can be obtained by using SPC. These process improvements include uniformity of output, reduced rework, fewer defective products, increased output, increased profitability, lower average cost, fewer errors, higher quality output, less scrapped cost, less machine downtime, less waste in production, increased job satisfaction and improved competitive position (Smith, 1991).

The basic techniques in SPC include the use of control charts to achieve and maintain statistical control in all phases of the process and in performing process capability studies in relation to product specifications and customer demands (Smith, 1991).

SPC consists of seven problem solving tools which can be considered useful in obtaining process stability and improving capability through the reduction of variability. These tools are known as the "Magnificent Seven", which comprise the histogram, check sheet, Pareto chart, cause and effect diagram, scatter diagram, defect concentration diagram and control charts (Montgomery, 2005).

A control chart which is a primary tool used in SPC is a graphical display of a certain descriptive statistics for specific quantitative measurement of the manufacturing process. Several different descriptive statistics can be used in a

control chart. There are two types of control charts, namely, control charts for variables data and control charts for attributes data.

1.2 Control Charts

The most powerful tool in SPC is the control chart. The general idea of a control chart originated from Shewhart of the Bell Laboratories in 1924 (Ryan, 2000). The construction of a control chart depends on the assumption of a certain statistical distribution. When used in the monitoring of a manufacturing process (or a non manufacturing process), a control chart can indicate whether a process is in-control or out-of-control. Ideally, we would want to detect an out-of-control situation as soon as possible after its occurrence. Also, we would like to have as few false alarms as possible (Ryan, 2000).

A control chart is a time sequence plot with "decision lines" added. The decision lines are the lower control limit (LCL), the center line (CL) and the upper control limit (UCL). These decision lines are chosen so that an out-of-control signal can be identified (Ryan, 2000). As long as all the sample points plot within the control limits, a process is assumed to be in-control and no action is necessary. However, sample points that plot beyond the control limits indicate that a process is out-of-control and investigations and corrective actions are required to find and remove the assignable causes responsible for this behaviour. The sample points on a control chart are usually connected with straight-line segments so that it is easier to visualize how the sequence of points has evolved overtime (Montgomery, 2005).

1.2.1 Univariate Control Charts

Traditionally, control charting techniques put great emphasis on the monitoring of shifts in the process mean (Huang and Chen, 2005). The most common

univariate variables control charting techniques used in the monitoring of shifts in the process mean are the Shewhart $\bar{X}$, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts. The $\bar{X}$ chart is quick in detecting large shifts while both the CUSUM and EWMA charts are sensitive to small shifts. One possible way to enhance the sensitivity of the Shewhart $\bar{X}$ chart towards small and moderate shifts is to use runs rules.

To date, numerous works on runs rules for the $\bar{X}$ chart have been suggested in the literature. Hurwitz and Mathur (1992) proposed a simple 2-of-2 rule with control limits having a width of 1.5 standard deviations from the center line. Klein (2000) suggested the 2-of-2 and 2-of-3 rules based on the Markov chain approach, where the control limits can be adjusted to give a desired in-control ARL. Khoo (2003) proposed the designs of the 2-of-4, 3-of-3 and 3-of-4 runs rules using the Markov chain approach. Khoo and Khotrun (2006) presented the improved 2-of-2 and 2-of-3 rules which increase the sensitivity of the standard 2-of-2 and 2-of-3 rules suggested by Klein (2000), in the detection of moderate and large shifts. Antzoulakos and Rakitzis (2008) proposed the revised *m*-of-*k* runs rules that demonstrated an improved performance in the detection of small to moderate shifts while maintaining the same superiority in detecting large shifts.

In addition to the use of runs rules, a synthetic $\bar{X}$ chart can also be used to increase the speed of the $\bar{X}$ chart in detecting small and moderate shifts in the mean. The synthetic $\bar{X}$ chart integrates the standard $\bar{X}$ chart and the conforming run length (CRL) chart. The synthetic $\bar{X}$ chart was suggested by Wu and Spedding (2000a). They showed that a synthetic $\bar{X}$ chart provides smaller out-of-control ARL than the standard $\bar{X}$ chart with or without runs rules, for any level of a mean shift. The synthetic $\bar{X}$ chart is also superior to the EWMA chart when the size of a shift in

the mean is greater than 0.8σ. Other works on the univariate synthetic control charts for the monitoring of the process mean are as follows: Wu and Spedding (2000b) presented a computer program that allows the computation of the upper and lower control limits based on a desired size of a shift in the mean. Wu et al. (2001) proposed a synthetic control chart for attributes data for the detection of increases in the fraction nonconforming. Wu and Yeo (2001) provided a C program to determine the control parameters and to calculate the average time to signal for the synthetic control chart suggested by Wu *et al.* (2001). Calzada and Scariano (2001) investigated the robustness to non-normality of the synthetic chart for monitoring the process mean. Davis and Woodall (2002) presented a Markov Chain model of the synthetic chart and used it to evaluate the zero-state and steady-state average run length (ARL) performances. Scariano and Calzada (2003) developed a synthetic chart for detecting decreases in the exponential mean, which combines the Shewhart chart for individuals and the CRL chart. Costa and Rahim (2006) presented a synthetic control chart for a joint monitoring of both the process mean and variance.

The synthetic control charts for variables data and the other commonly used control charting techniques such as the $\bar{X}$, EWMA and CUSUM charts all depend on the assumption that the distribution of a quality characteristic is normal or approximately normal. When the underlying distribution is nonnormal, three approaches are presently employed to deal with this problem. The first approach is to increase the sample size until the sample mean is approximately normally distributed. The second approach is to transform the original data so that the transformed data have an approximate normal distribution. The third approach is to use heuristic methods to design control charts, such as the $\bar{X}$ and R charts based on the weighted variance (WV) method proposed by Bai and Choi (1995), the $\bar{X}$, EWMA and

CUSUM charts based on the weighted standard deviation (WSD) method suggested by Chang and Bai (2001), and the $\bar{X}$ and R charts based on the skewness correction (SC) method presented by Chan and Cui (2003). Other works that deal with univariate control charts for skewed distributions include that of Wu (1996), Castagliola (2000), Nichols and Padgett (2005), Tsai (2007), Dou and Sa (2002), Chen (2004), and Yourstone and Zimmer (1992).

1.2.2 Multivariate Control Charts

The control charts mentioned in Section 1.2.1 deal with the controlling of only one quality characteristic. However, in many situations, we largely deal with two or more related quality characteristics (Mitra, 1998).

The problem of process monitoring involving several related variables of interest are sometimes called the multivariate quality control problem. The work on multivariate quality control was originally made by Hotelling in 1947, who applied his procedure to bombsight data during World War II. The topic on multivariate control charts is particularly important today as the automatic inspection procedure makes it relatively easy to measure many parameters on each unit of a product manufactured. The quality of many chemical and semiconductor processes are determined by several related variables. Because monitoring or analyses of these data with univariate SPC procedures are often ineffective and misleading, the use of multivariate methods have increased greatly in recent years (Montgomery, 2005).

The most widely used multivariate control charts are the Hotelling's T^2, multivariate CUSUM (MCUSUM) and multivariate EWMA (MEWMA) charts. The Hotelling's T^2 chart is based on only the current observation, whereas the MCUSUM and MEWMA charts accumulate information from the past observations.

The charting statistics evaluated by accumulating observations make the MCUSUM and MEWMA charts more sensitive in detecting small and moderate shifts in the mean vector of a multivariate process than the Hotelling's T^2 chart (Montgomery, 2005). A method to enhance the sensitivity of the Hotelling's T^2 chart in detecting small and moderate shifts is by using runs rules. Khoo and Quah (2003) proposed the use of the 2-of-2, 2-of-3 and 2-of-4 rules and presented a simple and effective approach of incorporating them into the Hotelling's T^2 chart. Aparisi et al. (2004) investigated the performance of the Hotelling's T^2 chart with runs rules and suggested the use of several rules by dividing the Hotelling's T^2 chart into attention zones and zones above and below the mean. Khoo *et al.* (2005) suggested the combined 2-of-2 and 1-of-1, 2-of-3 and 1-of-1, and 2-of-4 and 1-of-1 rules to enhance the performance of the Hotelling's T^2 chart based on the discussion in Khoo and Quah (2003). Koutras et al. (2006) introduced a run related chi-square control chart which signals an out-of-control process when k consecutive values of the test statistic exceed an appropriate upper control limit.

Besides the use of runs rules, the sensitivity of the Hotelling's T^2 chart towards small and moderate shifts in the mean vector can also be enhanced by implementing a synthetic T^2 control chart. The synthetic T^2 chart was suggested by Ghute and Shirke (2008a). A synthetic T^2 chart consists of a combination of the Hotelling's T^2 chart and the conforming run length (CRL) chart. The synthetic T^2 chart is an extension of the univariate synthetic $\overline{X}$ chart of Wu and Spedding (2000a).

Like the univariate control charts, a potential setback of multivariate control charts is the multivariate normality assumption of the underlying process distribution. In practice, the normality assumption is usually violated. For example, measurements

from chemical, filling and semiconductor processes are often skewed (Chang, 2007). Subsequently, a few multivariate control charting methods have been suggested in the literature to address this problem. These include the heuristic methods for the Hotelling's T^2 (Chang and Bai, 2004), MEWMA and MCUSUM (Chang, 2007) charts based on the weighted standard deviation (WSD) approach.

1.3 Objectives of the Study

The charting approaches of most univariate and multivariate control charts are based on the normality assumption. In real life situations, the normality assumption is usually violated. This thesis provides useful extensions of the univariate and multivariate synthetic charts for skewed distributions. The objectives of this thesis are as follows:

(i) To propose a more sensitive univariate synthetic variables control chart for skewed distributions in the detection of moderate and large shifts in the process mean. This new chart is called the synthetic WV$-\overline{X}$ chart hereafter. It is based on the idea of integrating the weighted variance (WV) method of Bai and Choi (1995) and the synthetic control charting approach of Wu and Spedding (2000a).

(ii) To propose the multivariate synthetic WSD$-T^2$ control chart. This chart is based on the idea of integrating the weighted standard deviation (WSD) method of Chang and Bai (2004) and the multivariate synthetic T^2 control charting approach of Ghute and Shirke (2008a).

1.4 Methodologies of the Study

In this thesis, for the univariate case, the weighted variance (WV) method of Bai and Choi (1995) is incorporated into the standard synthetic $\bar{X}$ chart of Wu and Spedding (2000a), leading to the proposed synthetic WV$-\bar{X}$ chart for monitoring the process mean of skewed distributions. The procedure used for the standard synthetic $\bar{X}$ chart of Wu and Spedding (2000a) is used to determine the optimal parameters for the proposed synthetic WV$-\bar{X}$ chart. Numerical integration is employed in the computation of the control chart's constant, d_2'. The performance of the synthetic WV$-\bar{X}$ chart, in terms of its false alarm and mean shift detection rates is evaluated via a Monte Carlo simulation using SAS version 9.

For the multivariate case, the weighted standard deviation (WSD) method of Chang and Bai (2004) and the standard multivariate synthetic T^2 control charting approach of Ghute and Shirke (2008a) are combined to form the proposed multivariate synthetic WSD$-T^2$ control chart for monitoring the process mean vector of skewed distributions. The approach used in the design of the standard multivariate synthetic T^2 chart of Ghute and Shirke (2008a) is used to determine the optimal limits and to compute the charting statistic of the proposed multivariate synthetic WSD$-T^2$ chart. A Monte Carlo simulation using SAS version 9 is conducted to evaluate the performance of the proposed multivariate synthetic WSD$-T^2$ chart, in terms of its false alarm and mean shift detection rates for six different directions of shifts.

1.5 Organization of the Thesis

This thesis is organized in the following manner: Chapter 1 introduces the objectives and methodologies of the study. It also gives some discussions on univariate and multivariate variables control charts and the statistical process control techniques. In Chapter 2, the univariate skewed distributions used in the later chapters are discussed, together with the normal distribution. A brief discussion on the Shewhart $\bar{X}_R$ and $\bar{X}_S$ charts is also given in this chapter together with a review on the univariate synthetic chart and the univariate charts for skewed distributions. In Chapter 3, the multivariate skewed distributions that are considered in the later chapters are discussed. The multivariate normal distribution is also discussed here. A discussion on the Hotelling's T^2, MCUSUM and MEWMA charts together with their extensions, as well as the multivariate synthetic charts and multivariate charts for skewed distributions are also presented in this chapter. Chapter 4 gives a detailed discussion on the proposed univariate synthetic $\mathrm{WV}-\bar{X}$ control chart. A performance evaluation of the synthetic $\mathrm{WV}-\bar{X}$ chart and an example on how it is put to work in a real situation are also described here. Chapter 5 presents the proposed multivariate synthetic $\mathrm{WSD}-T^2$ control chart together with a performance evaluation and an example to illustrate how the chart is constructed. Finally, conclusions for the thesis and suggestions for further research are presented in Chapter 6.

CHAPTER 2
A REVIEW ON UNIVARIATE DISTRIBUTIONS AND UNIVARIATE CONTROL CHARTS

2.1 Introduction

In this chapter, several continuous distributions that are important in statistical quality control and used in the present study will be discussed. These distributions include the normal distribution and skewed distributions, such as the lognormal, Gamma, Weibull and Burr distributions.

A quality characteristic that is measured on a numerical scale is called a variable, which can be length, width, temperature or volume. The Shewhart control charts for variables data which include the $\bar{X}_R$ and $\bar{X}_S$ charts that are widely used to monitor the mean of a process under the normality assumption will also be presented in this chapter.

One possible enhancement of the Shewhart control chart in increasing its speed in detecting shifts in the process mean is to construct a synthetic control chart. This chapter will discuss in detail the synthetic control chart.

In many situations, the normality assumption is usually violated. For example, the distributions of measurements from chemical and semiconductor processes are often skewed. Control charts for skewed distributions, such as the weighted variance $\bar{X}$ $\left(\text{WV}-\bar{X}\right)$, weighted standard deviation $\bar{X}$ $(\text{WSD}-\bar{X})$, WSD–CUSUM, WSD–EWMA and skewness correction $\bar{X}$ $(\text{SC}-\bar{X})$ charts will also be discussed in this chapter.

2.2 Univariate Distributions

2.2.1 The Normal Distribution

The normal distribution is one of the most important continuous distributions that is used in statistical quality control. The probability density function (pdf) of the normal distribution is

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ for } -\infty<x<\infty. \tag{2.1}$$

The mean and variance of the normal distribution are μ $(-\infty<\mu<\infty)$ and σ^2 $(\sigma^2>0)$, respectively (Montgomery, 2005). The cumulative distribution function (cdf) of the normal distribution can be defined as the probability in which the normal random variable X is less than or equal to some value, a, i.e.,

$$F(a)=\int_{-\infty}^{a}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx. \tag{2.2}$$

Because this definite integral cannot be evaluated in a closed form, the use of the transformation $z=\frac{x-\mu}{\sigma}$ is employed here. This transformation leads to the standard normal distribution (Montgomery, 2005). Figure 2.1 shows the pdf of a standard normal distribution.

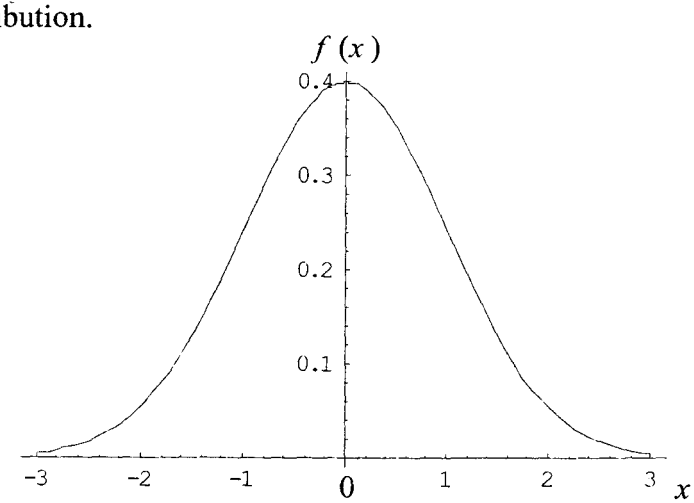


Figure 2.1 pdf of a standard normal distribution

The normal distribution has the following features and properties:

a) The normal distribution is bell-shaped.

b) The normal distribution is unimode.

c) The normal distribution is symmetric about the mean.

d) The total area under the pdf of the normal distribution is one.

f) A simple interpretation of the standard deviation, σ of the normal distribution is as follows (Montgomery, 2005):

(i) 68.26% of the population values fall within the limits defined by $\mu \pm 1\sigma$.

(ii) 95.46% of the population values fall within the limits defined by $\mu \pm 2\sigma$.

(iii) 99.73% of the population values fall within the limits defined by $\mu \pm 3\sigma$.

2.2.2 The Lognormal Distribution

A distribution that is related to the normal distribution is the lognormal distribution. Specifically, if $W = \log(X)$, i.e., the natural logarithm of X is normally distributed, then X follows a lognormal distribution. A general case of the lognormal distribution is the three-parameter lognormal distribution with pdf given by (Johnson and Kotz, 1970)

$$f(x) = \frac{1}{(x-\delta)\omega\sqrt{2\pi}} \exp\left[\frac{\{\log(x-\delta)-\theta\}^2}{2\omega^2}\right], \quad \text{for } x > \delta. \tag{2.3}$$

The corresponding cdf of this lognormal distribution is

$$F(x) = \Phi\left[\frac{\log(x-\delta)-\theta}{\omega}\right], \quad \text{for } x > \delta. \tag{2.4}$$

Here, $\Phi(\cdot)$ denote the standard normal cdf. In many applications, δ is considered to be zero, so that X is a positive random variable. This important case leads to the two-parameter lognormal distribution with parameters θ and ω. The

pdf of the two-parameter lognormal distribution is as follows (Johnson and Kotz, 1970):

$$f(x)=\frac{1}{x\omega\sqrt{2\pi}}\exp\left[\frac{\{\log(x)-\theta\}^2}{2\omega^2}\right],\quad \text{for } x>0. \tag{2.5}$$

Its cumulative distribution function (cdf) is

$$F(x)=\Phi\left[\frac{\log x-\theta}{\omega}\right],\quad \text{for } x>0. \tag{2.6}$$

The parameters, θ and ω^2 are the mean and variance of the normal random variable *W*. Moreover, the mean and variance of the lognormal distribution are functions of these parameters (Montgomery, 2005). The shape of the pdf of the lognormal distribution is based on the values of these parameters. A Mathematica program is written to plot the different shapes of the lognormal pdfs for selected values of θ and ω^2 (see Figure 2.2). Unlike the normal distribution, the lognormal distribution is not symmetric, but when ω is small (say, $\omega<1$), the lognormal distribution will be close to the normal distribution (Ryan, 2000).

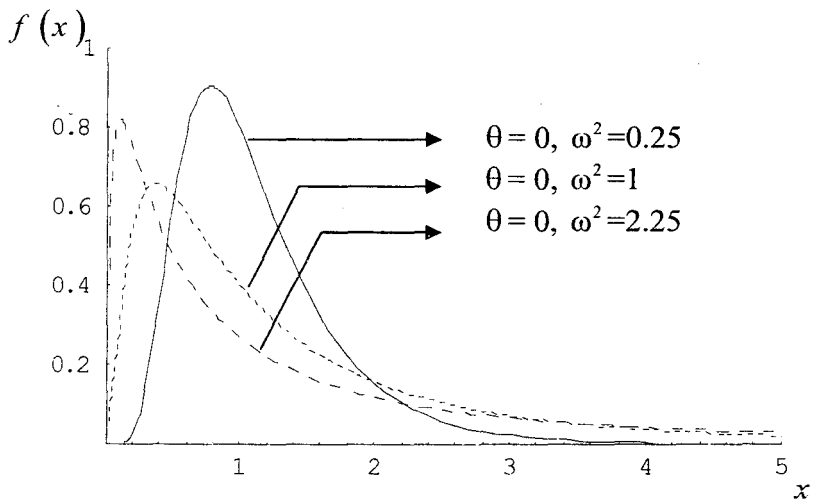


Figure 2.2. Two-parameter lognormal pdfs with $\theta=0$ for selected values of ω^2

The following are the moments and some properties of the two-parameter lognormal distribution (Johnson and Kotz 1970):

a) The rth moment of X about zero is

$$E\left(X^r\right)=\exp\left(r\theta+\frac{1}{2}r^2\omega^2\right). \tag{2.7}$$

b) The mean and variance of the two-parameter lognormal random variable are

$$E\left(X\right)=\exp\left(\theta+\frac{\omega^2}{2}\right) \tag{2.8}$$

and

$$\mathrm{Var}\left(X\right)=\exp\left(2\theta+\omega^2\right)\left[\exp\left(\omega^2\right)-1\right], \tag{2.9}$$

respectively.

c) The skewness and kurtosis coefficients of the two-parameter lognormal random variable are

$$\alpha_3=\left[\exp\left(\omega^2\right)+2\right]\sqrt{\exp\left(\omega^2\right)-1} \tag{2.10}$$

and

$$\alpha_4=\left[\exp\left(\omega^2\right)\right]^4+2\left[\exp\left(\omega^2\right)\right]^3+3\left[\exp\left(\omega^2\right)\right]^2-3, \tag{2.11}$$

respectively. Note that α_3 and α_4 do not depend on θ.

2.2.3 The Gamma Distribution

The pdf of a 3-parameter gamma distribution is defined as follows (Johnson and Kotz, 1970):

$$f\left(x\right)=\frac{\left(x-\gamma\right)^{\alpha-1}}{\eta^{\alpha}\Gamma\left(\alpha\right)}\exp\left[-\frac{\left(x-\gamma\right)}{\eta}\right], \text{ for } \alpha>0,\eta>0,\ x>\gamma. \tag{2.12}$$

It depends on three-parameters α, η and γ. These parameters are called the shape, scale and location parameters, respectively. The standard form of the gamma

distribution is obtained by putting $\eta = 1$ and $\gamma = 0$. Subsequently, the pdf will become as follows:

$$f(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \text{ for } x \geq 0. \tag{2.13}$$

The cdf of the gamma random variable corresponding to Equation (2.13) is given as (Johnson and Kotz, 1970)

$$F(x) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha)}, \text{ for } x \geq 0 \text{ and } \alpha > 0, \tag{2.14}$$

where $\Gamma_x(\alpha) = \int_0^x m^{\alpha-1}e^{-m}dm$ and $\Gamma(\alpha) = \int_0^\infty m^{\alpha-1}e^{-m}dm$. Note that if $\alpha = 1$, the gamma pdf in Equation (2.13) reduces to the exponential pdf with parameter one. A Mathematica program is written to plot the pdfs of the gamma distribution for $\eta = 1$ and $\alpha = 1$, 2 and 3 (see Figure 2.3).

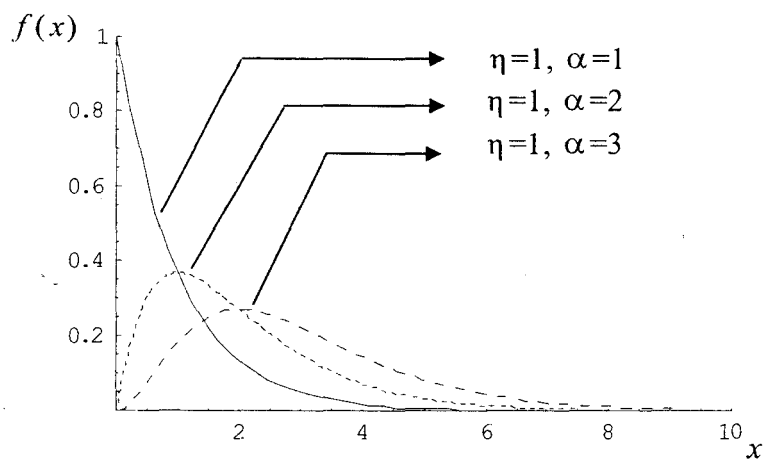


Figure 2.3. Gamma pdfs with $\eta = 1$ and $\alpha = 1$, 2 and 3

The following are the moments and some properties of the gamma distribution with pdf given in Equation (2.13) (Johnson and Kotz, 1970):

a) The rth moment of X about zero is

$$E(X^r) = \frac{\int_0^\infty x^{\alpha+r-1}e^{-x}dx}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}. \tag{2.15}$$

b) The mean and variance of X is

$$E(X) = \mathrm{Var}(X) = \alpha. \tag{2.16}$$

c) The distribution of X has a single mode at $x = \alpha - 1$ if $\alpha \geq 1$. The skewness and kurtosis coefficients of this gamma distribution are

$$\alpha_3 = 2\alpha^{-\frac{1}{2}} \tag{2.17}$$

and

$$\alpha_4 = 3 + \frac{6}{\alpha}, \tag{2.18}$$

respectively.

2.2.4 The Weibull Distribution

The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering. The pdf of the Weibull distribution, assuming that the location parameter is zero is as follows (Johnson and Kotz, 1970):

$$f(x) = \lambda'\beta(\lambda' x)^{\beta-1} e^{-(\lambda' x)^{\beta}}, \quad \text{for} \quad x > 0, \tag{2.19}$$

where $\lambda' > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. The cdf of the Weibull distribution is

$$F(x) = 1 - \exp\left[-(\lambda' x)^{\beta}\right], \quad \text{for} \quad x > 0. \tag{2.20}$$

The shape of the Weibull pdf is based on the appropriate selection of the parameters λ' and β. A Mathematica program is written to plot several Weibull pdfs for $\lambda' = 1$ and $\beta = 0.5$, 1, 2 and 4 (see Figure 2.4).

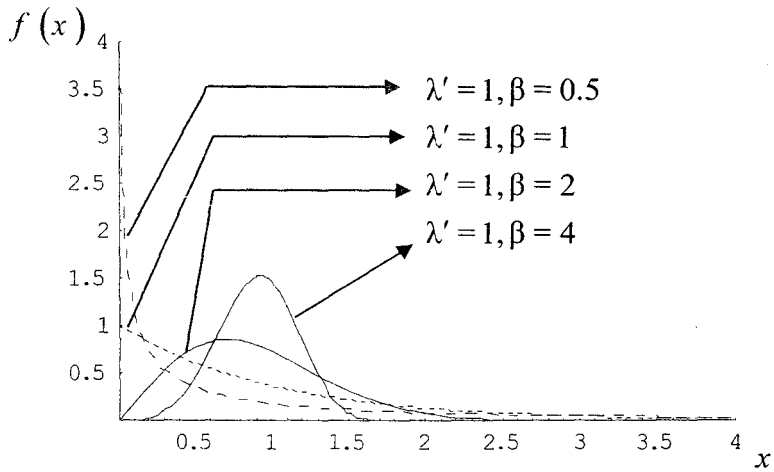


Figure 2.4. Weibull pdfs with $\lambda' = 1$ and $\beta = 0.5$, 1, 2 and 4

When $\beta = 1$, the Weibull distribution reduces to an exponential distribution. The moments and some properties of the Weibull distribution will be discussed here when the scale parameter $\lambda' = 1$, which leads to the standard Weibull distribution with the pdf

$$f(x) = \beta x^{\beta-1} e^{-x^{\beta}}, \quad \text{for } x > 0 \tag{2.21}$$

and cdf

$$F(x) = 1 - \exp(-x^{\beta}), \quad \text{for } x > 0. \tag{2.22}$$

For this case, the distribution of X depends only on the shape parameter β. Also the moments, coefficient of variation, skewness and kurtosis all depend only on the shape parameter β (Johnson and Kotz, 1970).

a) The rth moment about zero is

$$E(X^r) = \Gamma\left(\frac{r}{\beta} + 1\right). \tag{2.23}$$

b) The mean and variance of the Weibull distribution are

$$E(X) = \Gamma\left(\frac{1}{\beta} + 1\right) \tag{2.24}$$

and

$$\mathrm{Var}(X)=\Gamma\left(\frac{2}{\beta}+1\right)-\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2, \tag{2.25}$$

respectively.

c) The skewness and kurtosis coefficients of the Weibull distribution are as follows (Prabhakar *et al.*, 2004):

$$\alpha_3=\frac{\Gamma\left(\frac{3}{\beta}+1\right)-3\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{2}{\beta}+1\right)+2\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^3}{\left[\Gamma\left(\frac{2}{\beta}+1\right)-\left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]^{3/2}} \tag{2.26}$$

and

$$\alpha_4=\frac{k(\beta)}{\left[\Gamma\left(\frac{2}{\beta}+1\right)-\left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]^2}, \tag{2.27}$$

respectively, where

$$k(\beta)=-6\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^4+12\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\Gamma\left(\frac{2}{\beta}+1\right)-3\left[\Gamma\left(\frac{2}{\beta}+1\right)\right]^2$$
$$-4\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{3}{\beta}+1\right)+\Gamma\left(\frac{4}{\beta}+1\right).$$

2.2.5 The Burr Distribution

The pdf of the Burr distribution (Burr type XII) is (Rodriguez, 1977)

$$f(x)=bcx^{c-1}\left(1+x^c\right)^{-(b+1)},\ \text{for}\ x\geq 0,\ b>1\ \text{and}\ c>1 \tag{2.28}$$

and its cdf is

$$F(x)=1-\left(1+x^c\right)^{-b},\ \text{for}\ x>0,\ b>1\ \text{and}\ c>1, \tag{2.29}$$

where both b and c are the shape parameters of the distribution. The skewness and kurtosis of the Burr distribution exist if $bc > 3$ and $bc > 4$, respectively. The shape of the Burr pdf depends on the parameters b and c. A Mathematica program is written here to plot several Burr pdfs shown in Figure 2.5 for (b, c) = (2, 3), (3, 7) and (5, 1.2), respectively.

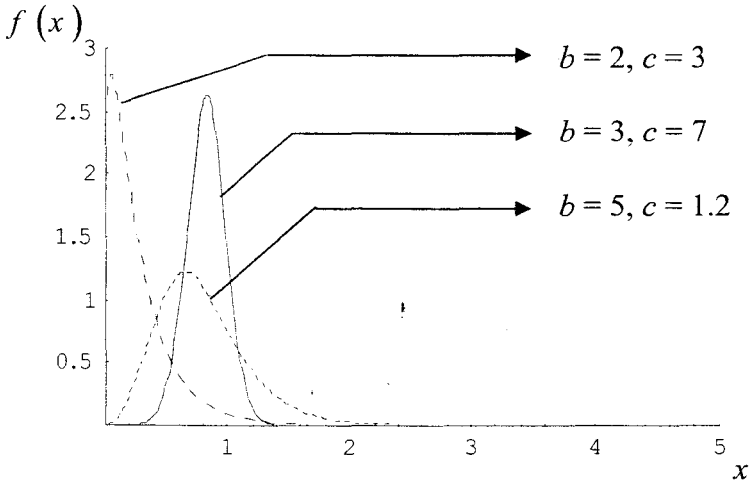


Figure 2.5. Burr pdfs for selected values of (b, c)

The moments and some properties of the Burr distribution (Burr type XII) are as follows (Tadikamalla, 1980):

a) The rth moment of X about zero is

$$E\left(X^r\right) = \frac{b\Gamma\left(b-\frac{r}{c}\right)\Gamma\left(\frac{r}{c}+1\right)}{\Gamma(b+1)}, \quad bc > r. \tag{2.30}$$

b) The mean and variance of X are

$$E(X) = \frac{\Gamma\left(1+\frac{1}{c}\right)\Gamma\left(b-\frac{1}{c}\right)}{\Gamma(b)} \tag{2.31}$$

and

$$\mathrm{Var}(X) = \frac{\Gamma\left(1+\frac{2}{c}\right)\Gamma\left(b-\frac{2}{c}\right)}{\Gamma(b)} - \left[\frac{\Gamma\left(1+\frac{1}{c}\right)\Gamma\left(b-\frac{1}{c}\right)}{\Gamma(b)}\right]^2, \tag{2.32}$$

respectively.

c) The skewness and kurtosis coefficients of X are (Rodriguez, 1977)

$$\alpha_3 = \frac{\left[\Gamma(b)\right]^2 \lambda_3 - 3\Gamma(b)\lambda_2\lambda_1 + 2\lambda_1^3}{\left\{\Gamma(b)\lambda_2 - \lambda_1^2\right\}^{3/2}} \tag{2.33}$$

and

$$\alpha_4 = \frac{\left[\Gamma(b)\right]^3 \lambda_4 - 4\left[\Gamma(b)\right]^2 \lambda_3\lambda_1 + 6\Gamma(b)\lambda_2\lambda_1^2 - 3\lambda_1^4}{\left\{\Gamma(b)\lambda_2 - \lambda_1^2\right\}^{3/2}}, \tag{2.34}$$

respectively, where

$$\lambda_j = \Gamma\left(\frac{j}{c}+1\right)\Gamma\left(b-\frac{j}{c}\right), \text{ for } j = 1, 2, 3 \text{ and } 4.$$

d) The density function given in Equation (2.28) is unimodal at $x = \frac{(c-1)}{(bc+1)^{1/c}}$ if $c > 1$ and L-shaped if $c \leq 1$ (Tadikamalla, 1980).

2.3 The Shewhart Control Charts

The idea of using control charts to monitor process data was developed by Walter A. Shewhart of the Bell Telephone Laboratories (Montgomery, 2005). The Shewhart control chart is based on the assumption that the distribution of the quality characteristic is normal or approximately normal.

The Shewhart control chart consists of three lines, the upper control limit, UCL, the center line, CL, and the lower control limit, LCL. These UCL and LCL are chosen so that the state of a process can be determined.

There are two types of Shewhart control charts which are classified according to the type of data of the underlying process. These charts are

(a) Shewhart control charts for variables

This type of control charts are used when the quality characteristic of a process is measured in a numerical scale (Montgomery, 2005).

(b) Shewhart control charts for attributes

This type of control charts are used when the quality characteristic of a process is measured by counting the number of defective items in a sample. In this thesis, we will discuss only the variables Shewhart control charts for monitoring the mean of a process, namely the Shewhart $\bar{X}_R$ and $\bar{X}_S$ control charts.

2.3.1 The $\bar{X}_R$ Chart

Assume that a process follows a normal distribution with in-control mean μ_X, and standard deviation, σ_X, where both μ_X and σ_X are known values. The control limits of the Shewhart $\bar{X}_R$ chart are as follows (Montgomery, 2005):

$$\text{UCL} = \mu_X + 3\frac{\sigma_X}{\sqrt{n}}, \tag{2.35a}$$

$$\text{CL} = \mu_X \tag{2.35b}$$

and

$$\text{LCL} = \mu_X - 3\frac{\sigma_X}{\sqrt{n}}. \tag{2.35c}$$

If the process parameters μ_X and σ_X are unknown, they are estimated from an in-control historical data set consisting of m samples, each of size, n. Let X_{ij} denotes the j^{th} observation in the i^{th} sample, for i=1, 2,..., m and j =1, 2,..., n. Also, let $\bar{X}_i$ and R_i represent the i^{th} sample mean and sample range, respectively. If the

grand mean and the average sample range are defined as $\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_i$ and $\bar{R} = \frac{1}{m}\sum_{i=1}^{m} R_i$, respectively, then the control limits of the Shewhart $\bar{X}$ chart when parameters are unknown are defined as follows (Montgomery, 2005):

$$\text{UCL} = \bar{\bar{X}} + 3\frac{\bar{R}}{d_2\sqrt{n}}, \tag{2.36a}$$

$$\text{CL} = \bar{\bar{X}} \tag{2.36b}$$

and

$$\text{LCL} = \bar{\bar{X}} - 3\frac{\bar{R}}{d_2\sqrt{n}}. \tag{2.36c}$$

The value of the control chart constant, d_2 which depends on the sample size, n, is given in most statistical quality control textbooks.

2.3.2 The $\bar{X}_S$ Chart

Although the $\bar{X}_R$ chart is widely used, it is occasionally desirable to estimate the process standard deviation directly, instead of indirectly through the use of the range R. This leads to the Shewhart $\bar{X}_S$ control chart. Generally, the $\bar{X}_S$ chart is preferred to its more familiar counterpart, the $\bar{X}_R$ chart when either (Montgomery, 2005)

(a) The sample size n is moderately large, i.e., n>10. The range method for estimating σ_X loses statistical efficiency for a moderate to large sample.

(b) The sample size, n is variable.

When the parameters μ_X and σ_X are both known, the control limits of the Shewhart $\bar{X}_S$ chart are similar to those of the Shewhart $\bar{X}_R$ chart given in Section 2.3.1.

However, when the parameters μ_X and σ_X are unknown, they can be estimated by analyzing past data. For this case, μ_X and σ_X are estimated as $\overline{\overline{X}}$ and $\frac{\overline{S}}{c_4}$, respectively, where $\overline{\overline{X}} = \frac{1}{m}\sum_{i=1}^{m} \overline{X}_i$ and $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$. Here, S_i denotes the i^{th} sample standard deviation. Then the control limits of the Shewhart $\overline{X}_S$ chart are computed as follows (Montgomery, 2005):

$$\text{UCL} = \overline{\overline{X}} + \frac{3\overline{S}}{c_4\sqrt{n}}, \tag{2.37a}$$

$$\text{CL} = \overline{\overline{X}} \tag{2.37b}$$

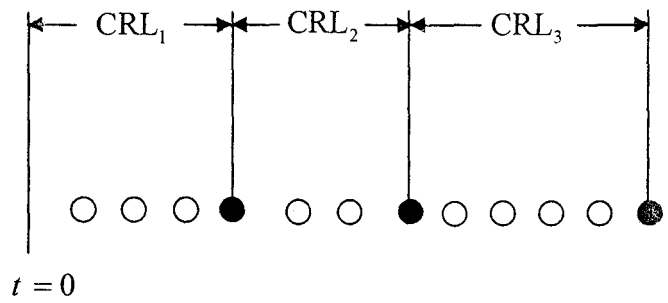
$$\text{LCL} = \overline{\overline{X}} - \frac{3\overline{S}}{c_4\sqrt{n}}. \tag{2.37c}$$

Here, the value of the control chart's constant, c_4 which depends on the sample size, n, is given in most statistical quality control textbooks.

2.4 The Synthetic $\overline{X}$ Control Chart

The synthetic $\overline{X}$ control chart was introduced by Wu and Spedding (2000a) to improve upon the performance of the Shewhart $\overline{X}$ control chart for detecting small and moderate shifts in the process mean. It also surpasses the exponentially weighted moving average (EWMA) and the joint $\overline{X}$ −EWMA charts, in terms of the detection power for detecting a mean shift of greater than 0.8σ. The synthetic $\overline{X}$ chart is based on the idea of integrating the Shewhart $\overline{X}$ chart and the conforming run length (CRL) chart. The synthetic $\overline{X}$ chart consists of the $\overline{X}/S$ sub-chart and the CRL/S sub-chart (Wu and Spedding, 2000a). Figure 2.6 illustrates how the CRL value, i.e., the number of inspected samples between two consecutive nonconforming

samples (inclusive of the ending nonconforming sample) is determined. Here, if we assume that a process starts at $t = 0$, then, $CRL_1 = 4$, $CRL_2 = 3$ and $CRL_3 = 5$.



○ Conforming sample

● Nonconforming sample

Figure 2.6. Conforming Run Length

The operation of a synthetic $\bar{X}$ chart is as follows (Wu and Spedding, 2000a):

Step 1 Determine the lower control limit, L, of the CRL/S sub-chart and compute the control limits, $UCL_{\bar{X}/S}$ and $LCL_{\bar{X}/S}$ of the $\bar{X}/S$ sub-chart using the following formulae:

$$UCL_{\bar{X}/S} = \mu_X + k\sigma_{\bar{X}} \tag{2.38a}$$

and

$$LCL_{\bar{X}/S} = \mu_X - k\sigma_{\bar{X}}, \tag{2.38b}$$

where μ_X denotes the in-control mean and $\sigma_{\bar{X}}$ is the standard deviation of the sample mean. A procedure and formulae to determine the optimal parameters, k and L, which give the minimum out-of-control (o.o.c.) ARL for the size of a shift, δ_d, where a quick detection is desired, based on an in-control ARL of interest are provided by Wu and Spedding (2000a). These procedure and formulae will also be discussed in Chapter 4. Note that Wu and