ROOTS OF POLYNOMIALS

by

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PUNCA POLINOMIAL

ABSTRAK

Punca polinomial mempunyai pelbagai applikasi dalam kehidupan kita. Ia timbul bukan sahaja dalam bidang sains dan kejuruteraan malah dalam bidang perniagaan dan ekonomi dan juga dalam bidang rekaan dan seni.

Dalam projek ini, kami mengkaji Teorem Asasi Aljabar dan pembuktiannya untuk mengesahkan kewujudan punca polinomial. Kami memerikan kaedah pencarian penyelesaian tepat bagi persamaan kuadratik dengan melengkapkan kuasa dua atau formula kuadratik, dengan geometri dan pemfaktoran. Algoritma Cardano dikaji untuk mendapatkan penyelesaian tepat bagi persamaan kubik dan pembahagian sintesis untuk memfaktorkan polinomial, khas bagi polinomial berdarjah tinggi. Kami membincang pendekatan berangka bagi punca hampiran polinomial dengan kaedah pembahagian duasama, kaedah Newton, kaedah secant dan algoritma hibrid. Kepentingan punca diilustrasikan dengan beberapa aplikasi.

ABSTRACT

Roots of polynomials have diverse applications in real life. It arises not only in the fields of science and engineering but also in the field of business and economics and even in the field of design and art.

In this project, we study the Fundamental Theorem of Algebra and present its proof to confirm the existence of roots of polynomials. We describe methods for finding the exact solutions of quadratic equations by completing the square or the quadratic formula, by geometry and factorization. Cardano's algorithm is studied to obtain the exact solutions for cubic equations and synthetic division for factorizing polynomials, in particular for polynomials of higher degree. We discuss the numerical approach for the approximate roots of polynomials by bisection method, Newton's method, secant method and the hybrid algorithm. The importance of the roots is illustrated by some examples.

CHAPTER 1

INTRODUCTION

1.1 Introduction of the Roots of Polynomials

A polynomial is built from terms called monomials, each of which consists of a coefficient multiplied by one or more variables. It is the sum of a finite number of monomials. In this project, we are considering only polynomials in one variable and with real numbers as coefficients. The variable may have a non-negative integer exponent that is the degree of that variable in that monomial.

The polynomial

$$3x^2 - 5x + 4$$

consists of three monomials, the first of degree two, the second of degree one and the third of degree zero. The degree of a polynomial is the largest degree of any one term. A polynomial of degree zero is called a constant, of degree one is said to be linear, of degree two is said to be quadratic, of degree three is said to be cubic. A polynomial of degree n in x has the general representation

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

where a_i, $0 \le i \le n$, are real numbers and $a_n \ne 0$.

A polynomial function is a function defined by evaluating a polynomial. For example, the function f defined by

$$f(x) = 2x^3 + 6x + 2$$

is a polynomial function. Evaluation of a polynomial consists of assigning a number to the variable and carrying out the indicated multiplications and additions.

A polynomial equation is an equation in which a polynomial is set equal to zero. The degree of the polynomial equation is the highest exponent of the variable in the equation. For example,

$$3x^2 + 4x - 5 = 0$$

and

$$-9x^{20} + 17x^{11} + x^2 - 10 = 0$$

are respectively polynomial equations of degree 2 and 20.

A real or complex number x_0 is a solution to the polynomial equation $P(x) = 0$ if it satisfies the equation. The solutions to the polynomial equation $P(x) = 0$ are called the roots of the polynomial $P(x)$ and they are the zeroes of the polynomial function and thus the x - intercepts of its graph if they are real.

Some polynomials, such as $f(x) = x^2 + 1$, do not have any roots among the real numbers. If, however, the set of allowed solutions is expanded to the complex numbers, then every non-constant polynomial has at least one distinct root. This follows from the Fundamental Theorem of Algebra. However, in this project we are emphasising more on the real roots of the polynomial.

Formulas for the roots of polynomials up to a degree of 2 have been known since ancient times. On clay tablets dated between 1800 BC and 1600 BC, the ancient

Babylonians left the earliest evidence of the discovery of quadratic equations, and also gave early methods for solving them. The Indian mathematician Baudhayana who wrote a Sulba Sutra in ancient India about 8th century BC first used quadratic equations of the form $ax^2 = c$ and $ax^2 + b = c$ and also gave methods for solving them.

Babylonian mathematicians from about 400 BC and Chinese mathematicians from about 200 BC used the method of completing the square to solve quadratic equations with positive roots, but did not have a general formula. Euclid produced a more abstract geometrical method around 300 BC.

In the early 16th century, the Italian mathematician Scipione del Ferro (1456 – 1526) found a method for solving cubic equations of the form $x^3 + mx = n$, where m, n are integers. Del Ferro kept his achievement a secret until just before his death, when he told his student Antonio Fiore about it.

In 1530, Niccolo Tartaglia (1500 – 1557) received questions in the form $x^3 + mx = n$, for which he had worked out a general method, won the contest between he and Antonio Fiore.

Later, Tartaglia was persuaded by Gerolamo Cardano (1501 – 1576) to reveal his secret for solving cubic equations. Tartaglia did so only on the condition that Cardano would never reveal it. A few years later, Cardano learned about Ferro's prior work and broke the promise by publishing Tartaglia's method in his book *Ars Magna* (1545) with credit given to Tartaglia.

Approximate solutions to any polynomial equation can be found either by Newton's method or by one of the many more modern methods of approximating solutions. Newton's method was described by Isaac Newton (1642 – 1727) in *De analysi* *per aequationes numero terminorum infinitas*, written in 1669 and published in 1711 by William Jones and in *De metodis fluxionum et serierum infinitarum*, written in 1671, translated and published as *Method of Fluxions* in 1736 by John Colson.

Fundamental Theorem of Algebra was conjectured as early as the sixteenth century. Several incorrect proofs of it were published before Carl Friedrich Gauss (1777 – 1855) found a satisfactory proof in 1797. The proof was mainly geometric, but it had a topological gap. A rigorous proof was published by Argand in 1806; it was here that, for the first time, the Fundamental Theorem of Algebra was stated for polynomials with complex coefficients, rather than just real coefficients. Gauss produced two other proofs in 1816 and another version of his original proof in 1849.

1.2 Objectives of the Project

The objectives of this project are:

- To study some of the theorems and their proofs involved in root-finding methods
- To study the methods for finding the exact roots of quadratic and cubic polynomials
- To study the common numerical methods of root-finding
- To illustrate the importance of root-finding with some applications.

Since this project is about roots of polynomials it is natural to first ask the question, "How do we know the existence of the roots?" If we are not assured of the existence of the roots, then this project would be meaningless. Hence, we present and prove the Fundamental Theorem of Algebra which confirms the existence of the roots of polynomials. In this project we study the methods for exact roots of quadratic and cubic polynomials. We describe numerical methods for approximate roots of polynomials. We also illustrate the importance of root-finding with several apllications.

1.3 Scope of the Project

The Scope of this master project includes the introduction of the roots of polynomials, the Fundamental Theorem of Algebra, the quadratic and cubic equations, numerical estimation for the roots of nonlinear polynomials and the applications of the roots of polynomials.

The discussion begins by introducing the roots of polynomials and a brief history of quadratic and cubic equations and the Newton's method. In Chapter 2, we state the Fundamental Theorem of Algebra. The proof of the Fundamental Theorem of Algebra involves some analysis, at the very least the concept of continuity of real or complex functions and we prove it by contradiction using Liouville's theorem.

The third chapter is mainly concerned with the quadratic equations and the various methods of finding its solutions. Synthetic division is also discussed in this chapter under the heading of factorization. Chapter 4 covers cubic polynomials and derives the Cardano's formula in finding the roots.

The next chapter is about the common numerical methods of root-finding, namely the bisection method, Newton's method and secant method. It also discusses the hybrid algorithm with fast global convergence. Chapter 6 focuses on the applications of the roots of polynomials.

The last chapter is the conclusion of the project about the roots of polynomials.

CHAPTER 2

THE FUNDAMENTAL THEOREM OF ALGEBRA

As this project concerns the roots of polynomial equations, we shall begin by discussing what is probably the most important result in the theory of equations, i.e. the Fundamental Theorem of Algebra (Beaumont and Pierce, 1963).

Theorem 2.1 (The Fundamental Theorem of Algebra)

If P is a non-zero complex polynomial of degree greater or equal to 1, then P has at least one root in $\mathbb{C}$, where $\mathbb{C}$ denotes the complex plane.

This theorem was conjectured as early as the sixteenth century. Several incorrect proofs of it were published before Gauss found a satisfactory proof in 1797. Gauss ultimately gave five different proofs of the Fundamental Theorem of Algebra, each of which introduced new ideas and methods, which have greatly influenced the development of mathematics.

Proving the theorem algebraically is quite difficult and it involves tools such as field, irreducible polynomial, subring and others, which are not covered in this project. However, we shall prove this theorem using Liouville's theorem (Marsden, 1973). Before proving Liouville's theorem, we shall state Cauchy's Inequalities, which will be used in the proof. We recall that a complex function f is said to be analytic on $A \subset \mathbf{C}$ if f is complex differentiable at each $z_0 \in A$. The phrase "analytic at z_0" means complex differentiable on a neighborhood of z_0. A function that is defined and analytic on the whole plane $\mathbf{C}$ is said to be entire.

Theorem 2.2 (Cauchy's Inequalities)

Let f be an analytic function on a region A in $\mathbf{C}$ and let γ be a circle with radius r and center z_0 that lies in A. Assume that the disk $\{z \in \mathbf{C} \mid |z - z_0| < r\}$ also lies in A. Suppose that $|f(x)| \le M$ for all z on γ. Then, for any $k = 0, 1, 2, \ldots,$

$$\left| f^{(k)}(z_0) \right| \le \frac{k!}{r^k} M.$$

Theorem 2.3 (Liouville's Theorem)

If f is entire and bounded for all $z \in \mathbf{C}$, then f is constant.

Proof:

Since f is bounded, there is an $M > 0$ such that $|f(z)| \le M$ for any $z \in \mathbf{C}$. For any $a \in \mathbf{C}$, Cauchy's inequality gives

$$|f'(a)| \le \frac{M}{r}$$

for any $r > 0$. By letting $r \to \infty$, we get

$$f'(a) = 0.$$

Thus $f'(z) = 0$ for all $z \in \mathbf{C}$ and therefore the function f is a constant function on $\mathbf{C}$.

After stating Liouville's theorem and proving it, we are now able to prove the Fundamental Theorem of Algebra. We shall prove by contradiction by assuming that the polynomial p has no zero in $\mathbf{C}$. We then prove that $f(z) = \frac{1}{p(z)}$ is a bounded entire function and use Liouville's theorem to get a contradiction (cs.usm.my/~vravi/mss301.html).

Proof: (The Fundamental Theorem of Algebra)

Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$$

be a polynomial of degree n when $n \geq 1$. Then $a_n \neq 0$. Note that as $|z| \to \infty$, then $|p(z)| \to \infty$. This follows as

$$|p(z)| = |z^n| \left| a_n + \frac{a_{n-1}}{z} + ... + \frac{a_0}{z^n} \right|$$

and when $|z| \to \infty$, $|z^n| \to \infty$ and $\left| a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \to |a_n|$.

Assume that $p(z) \neq 0$ for any $z \in \mathbf{C}$. We are going to show that $f(z) = \frac{1}{p(z)}$ is bounded. Let

$$A = \max\left(\frac{1}{2n}, \frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_n}, \ldots, \frac{a_0}{a_n} \right).$$

For $|z| \geq 2nA \geq 1$, we have

$$\left|\frac{a_{n-k}}{a_n z^k}\right| \leq \frac{A}{|z|^k} \leq \frac{A}{|z|} \leq \frac{1}{2n}$$

and therefore by applying the triangle inequality

$$\left|\frac{a_{n-1}}{a_n z} + \ldots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right| \leq \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = \frac{1}{2}.$$

By writing

$$p(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \ldots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right),$$

we see that, for $|z| \geq 2nA \geq 1$,

$$|p(z)| \geq |a_n||z|^n \left(1 - \left|\frac{a_{n-1}}{a_n z} + \ldots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right|\right)$$

$$\geq |a_n||z|^n \left(1 - \frac{1}{2}\right)$$

$$= \frac{|a_n||z|^n}{2}$$

$$\geq \frac{|a_n|(2nA)^n}{2}.$$

Hence $\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|(2nA)^n}$ for all $|z| \geq 2nA$. Since $p(z) \neq 0$ and it is continuous for any $z \in \mathbf{C}$, the function $\frac{1}{p(z)}$ is continuous in $\mathbf{C}$, and therefore on the compact set $|z| \leq 2nA$, it is bounded, say by M. Thus we have for all $z \in \mathbf{C}$,

$$\frac{1}{|p(z)|} \le \max\left\{M, \frac{2}{|a_n|(2nA)^n}\right\}.$$

Thus $\frac{1}{p(z)}$ is a bounded entire function and hence by Liouville's Theorem, it reduces to a constant function, which is a contradiction. This proves that there is at least one root for $p(z)$.

Corollary 2.1

Any polynomial equation $p(z)=0$ of degree $n>0$ has exactly n roots in $\mathbb{C}$.

Proof:

We prove the result by mathematical induction. Clearly any polynomial of degree 1 has exactly one root in $\mathbb{C}$. Assume that the result is true for any polynomial of degree m. Let $p(z)$ be a polynomial of degree $m+1$. By the Fundamental Theorem of Algebra, there is a root for $p(z)$ and denote the root by α. Then $(z-\alpha)$ is a factor of $p(z)$ and hence $p(z)=(z-\alpha)q(z)$ where $q(z)$ is a polynomial of degree m and hence it has exactly m roots. Thus $p(z)$ has exactly $m+1$ roots.

The general form of the degree n equation is:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0. \tag{2.1}$$

From Corollary 2.1, we know that any polynomial equation of the degree n as in (2.1) always has n solutions in $\mathbb{C}$. In particular cases, some or all of these n solutions could be equal to one another. Though the coefficients a_i may be real numbers, the solutions could be real or complex numbers.

CHAPTER 3

QUADRATIC EQUATIONS

A quadratic equation is a polynomial equation of the second degree. The general form is $ax^2 + bx + c = 0$, where a$\neq$0. a, b, and c are coefficients which are real.

Quadratic equations are called quadratus in Latin, which means 'squares'. In the leading term the variable is squared.

There are a few methods of finding the solutions of a quadratic equation $ax^2 + bx + c = 0$ or equivalently the roots of a polynomial $ax^2 + bx + c$. They are discussed below.

(http://faculty.ed.umuc.edu/~swalsh/Math%20Articles/Quadratic%20Equations.html)

3.1 Completing the Square

This method makes use of the algebraic identity: $x^2 + 2xy + y^2 = (x + y)^2$.

Dividing the quadratic equation

$$ax^2 + bx + c = 0 \tag{3.1}$$

by a gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Its equivalent is

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{3.2}$$

To complete the square is to find some constant k such that

$$x^2 + \frac{b}{a}x + k = x^2 + 2xy + y^2.$$

For this equation to hold it must be the case that $y = \frac{b}{2a}$ and $k = y^2$, thus $k = \frac{b^2}{4a^2}$.

Adding this constant to both sides of the equation (3.2) produces

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

The left is now a perfect square of $x + \frac{b}{2a}$. This gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides yields

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

So

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.3}$$

which is also known as quadratic formula.

From the formula (3.3), the term underneath the square root sign $b^2 - 4ac$ is called the discriminant of the quadratic equation.

The discriminant determines the number and nature of the roots. There are three cases:

i) If it is positive, there are two distinct roots, both of which are real numbers.

ii) If it is zero, there is exactly one root, and that root is a real number. Sometimes, it is called a double root, its value is $x = -\frac{b}{2a}$.

iii) If it is negative, there are no real roots but there two distinct complex roots. They are

$$x = \frac{-b}{2a} + i\left(\frac{\sqrt{4ac - b^2}}{2a}\right) \text{ and } x = \frac{-b}{2a} - i\left(\frac{\sqrt{4ac - b^2}}{2a}\right)$$

where i is the imaginary unit.

Thus, a quadratic equation (3.1) with real coefficient has two, not necessarily distinct roots, which may be real or complex, given by the quadratic formula (3.3), the symbol '$\pm$' indicates that both

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions.

3.2 The Method of Geometry

Consider the quadratic equation

$$ax^2 + bx + c = 0.$$

It is easy to see that the solutions are the intercept of x-axis of the quadratic function $f(x) = ax^2 + bx + c$ when these solutions are real.

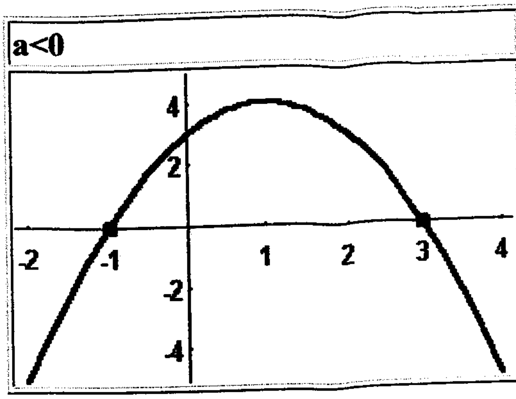

Figure 3.1
The quadratic function for $a < 0$

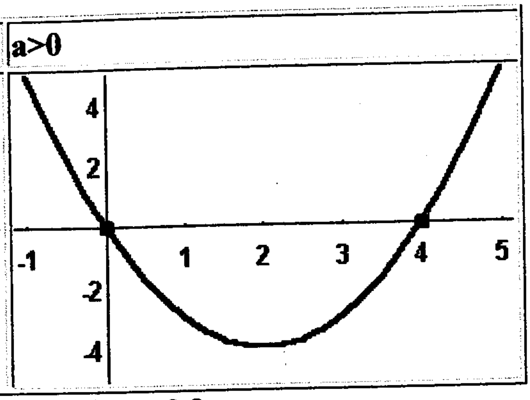

Figure 3.2
The quadratic function for $a > 0$

The two graphs, Figure 3.1 and Figure 3.2 are examples that show a real valued quadratic function, $f(x) = ax^2 + bx + c$, for a < 0 and a > 0, are parabolas.

If *a*, *b*, and *c* are real numbers, and the domain of *f* is the set of real numbers, then the zeros of *f* if exist, are exactly the *x*-coordinates of the points where the graph touches or intersects the *x*-axis. It follows from the above that, if the discriminant is positive, the graph intersects the *x*-axis at two points; if zero, the graph touches at a point; and if negative, the graph does not touch the *x*-axis.

(http://en.wikipedia.org/wiki/Quadratic_equation#Vi.C3.A8te.27s_formulaQuadratic)

3.3 Factorization

Before we proceed to factorization, let us present and prove the factor theorem (Smith,1986).

Theorem 3.1 (The Factor Theorem)

$(x - r)$ is a factor of a polynomial $P(x)$ if and only if r is a root of $P(x)$.

Proof:

Suppose $(x-r)$ is a factor of $P(x)$ then $P(r)$ will have the factor $(r-r)$, which is 0. This will make $P(r)=0$. This means that r is a root.

Conversely, if r is a root of $P(x)$, then $P(r)=0$. But according to the remainder theorem, $P(r)=0$ means that upon dividing $P(x)$ by $(x-r)$, the remainder is 0. Therefore $(x-r)$ is a factor of $P(x)$.

By the Fundamental Theorem of Algebra, the factor theorem allows us to conclude:

$$P(x)=a\,(x-r_n)(x-r_{n-1})\ldots(x-r_2)\,(x-r_1)$$

for some $a, r_1, r_2, \ldots r_n \in \mathbf{C}$.

To solve a quadratic equation by factoring, we may do the following simple steps:

(i) first we must make sure it is in standard form, $ax^2+bx+c=0$

(ii) then we must factor the left side if possible

(iii) lastly we set each of the two factors equal to zero and solve the equations we get to find the solutions to the equation.

Example 3.1

Solve the equation

$$2x^2+x-1=0.$$

Solution: Factoring yields the equation

$$(2x-1)(x+1)=0.$$

Hence, we have

$$2x-1=0 \quad or \quad x+1=0$$

which yield

$$x=\frac{1}{2} \quad or \quad x=-1.$$

Thus, $x=\frac{1}{2}$ *or* $x=-1$ are the roots of the equation $2x^2+x-1=0$.

Note that not all polynomials can be factored into real factors. For example the polynomial,

$$2x^2+3x-1,$$

cannot be factored easily. Thus, to find its root we have to use either by quadratic formula or the method of completing the square. The quadratic formula or completing the square are always applicable in solving quadratic equations and the solutions are exact. However, when we actually use these methods we generally have to approximate the numbers involved, especially when extracting square roots.

Now we are going to show a method of factorizing polynomials, in particular for polynomials of higher degree, by synthetic division (Smith, 1986).

Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form $x-s$. It can be used in place of the standard long division algorithm. This method reduces the polynomial and the linear factor into a set of numeric values. After these values are processed, the resulting set of numeric outputs is used to construct the polynomial quotient and the polynomial remainder.

Suppose that

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{3.4}$$

is a given polynomial, r is an exact solution of $P(x)$. Then $P(r) = 0$. One way to test whether s is indeed a root of $P(x)$ is to see whether $x - s$ divides $P(x)$. We therefore write

$$\frac{P(x)}{x-s} = Q(x) + \frac{b_0(s)}{x-s}$$

where $Q(x)$ is said to be the quotient which must be a polynomial of degree $n-1$ and b_0 is said to be the remainder which is a constant with respect to x but of course depends on s, as the notation implies. If s is an exact root then b_0 will be zero. These statements and the following presentation are valid whether s is real or complex number.

We have

$$P(x) = (x - s)\,Q(x) + b_0(s) \tag{3.5}$$

for all numbers s. We need to determine the coefficients of the degree $n-1$ polynomial $Q(x)$ and the remainder b_0, given the coefficients of $P(x)$. To determine them synthetically, let

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_2 x + b_1 .$$

Then

$$\begin{aligned} P(x) &= (x-s)\,Q(x) + b_0 \\ &= (x-s)\left(b_n x^{n-1} + \ldots + b_2 x + b_1\right) + b_0 \\ &= x\left(b_n x^{n-1} + \ldots + b_1\right) - s\left(b_n x^{n-1} + \ldots + b_1\right) + b_0 \\ &= b_n x^n + \left(b_{n-1} - s b_n\right) x^{n-1} + \ldots + \left(b_1 - s b_2\right) x + \left(b_0 - s b_1\right) \end{aligned} \tag{3.6}$$

should be identical to (3.4).

For polynomials to be equal as functions of x, every corresponding coefficient must be the same. Setting corresponding coefficients of (3.4) and (3.6) equal to one another, we get

$$a_n = b_n \ , \quad a_{n-1} = b_{n-1} - s\,b_n \ , \quad . \ . \ . \ , a_0 = b_0 - s\,b_1 \, .$$

To use these equations, we need to solve them for the $\{b_k\}$, one at a time, starting with $b_n = a_n$. We therefore write the equations in the form

$$b_n = a_n \ , \quad b_{n-1} = a_{n-1} + s\,b_n \ , \quad . \ . \ . \ , b_0 = a_0 + s\,b_1 \, .$$

This set of $n+1$ equations can be compactly written as

$$b_k = a_k + s\,b_{k+1} \tag{3.7}$$

with $k = n, n-1, \ldots, 1, 0$. Here b_{n+1} is assumed to be zero.

Equation (3.7) can be viewed as a finite iteration process. The first n numbers obtained are the coefficients of the quotient polynomial $Q(x)$, in order, starting with the highest degree term and the last number is the remainder b_0. It also follows from equation (3.5) that

$$b_0 = P(s)$$

so this process can be used to evaluate a polynomial. In particular, if $b_0 = 0$ then s is a root of the polynomial $P(x)$ and $P(x) = (x - b_0)(b_n x^{n-1} + \ldots + b_2 x + b_1)$.

The process of synthetic division is performed efficiently using the following pattern:

$$\begin{array}{r|cccccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & a_0 \\ + & & sb_n & sb_{n-1} & \cdots & sb_3 & sb_2 & sb_1 \\ \hline & b_n & b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 & \big| \; b_0 \end{array}$$

We shall illustrate how synthetic division works by looking at two examples.

Example 3.2:

Find the roots of $15x^4 + x^3 - 52x^2 + 20x + 16$.

Solution:

Let us try $x = 1$, i.e. put $s = 1$.

$$\begin{array}{llrrrrr} & & 15 & 1 & -52 & 20 & 16 \\ s=1 & + & & 15 & 16 & -36 & -16 \\ \hline & & 15 & 16 & -36 & -16 & \big| \; 0 \end{array}$$

This gives a zero remainder, so $x = 1$ is a root, which means that $(x-1)$ is a factor. Since we divide a linear factor out of $(x-1)$, this means that the quotient is a cubic:

$$15x^3 + 16x^2 - 36x - 16.$$

Thus we need to find another zero before we can apply the quadratic formula. We will try $x = -2$:

$$\begin{array}{llrrrr} & & 15 & 16 & -36 & -16 \\ s=-2 & + & & -30 & 28 & 16 \\ \hline & & 15 & -14 & -8 & \big| \; 0 \end{array}$$

Since we get a zero remainder, $x = -2$ is a root, so $(x+2)$ is a factor. This leaves us with the quadratic :

$$15x^2 - 14x - 8.$$

By factoring, we obtain

$$(3x-4)\,(5x+2).$$

Then the fully factored form is:

$$15x^4+x^3-52x^2+20x+16=(x-1)(x+2)(3x-4)(5x+2).$$

The roots are $x=1,-2,\frac{4}{3},\frac{-2}{5}$.

Example 3.3:

Given that $(2-i)$ is a root of $x^5-6x^4+11x^3-x^2-14x+5$, fully solve the equation $x^5-6x^4+11x^3-x^2-14x+5=0$.

Solution:

Since $(2-i)$ is a given root, we shall use synthetic division and divide out $(2-i)$:

$$\begin{array}{lcccccc|c} & & 1 & -6 & 11 & -1 & -14 & 5 \\ s=2-i & + & & 2-i & -9+2i & 6+2i & 12-i & -5 \\ \hline & & 1 & -4-i & 2+2i & 5+2i & -2-i & 0 \end{array}$$

Since $(2-i)$ is a root, then its complex conjugate $(2+i)$ is also a root of the equation.

$$\begin{array}{lccccc|c} & & 1 & -4-i & 2+2i & 5+2i & -2-i \\ s=2+i & + & & 2+i & -4-2i & -4-2i & 2+i \\ \hline & & 1 & -2 & -2 & 1 & 0 \end{array}$$

This leaves us with a cubic:

$$x^3-2x^2-2x+1.$$

Let us try $x=-1$:

$$\begin{array}{lcccc|c} & & 1 & -2 & -2 & 1 \\ s=-1 & + & & -1 & 3 & -1 \\ \hline & & 1 & -3 & 1 & 0 \end{array}$$

Now we get a quadratic:

$$x^2 - 3x + 1.$$

Thus by using quadratic formula, we get

$$x = \frac{3 \pm \sqrt{5}}{2}.$$

Hence, all the roots of $x^5 - 6x^3 + 11x^3 - x^2 - 14x + 5$ are

$$x = 2 \pm i, \quad \frac{3 \pm \sqrt{5}}{2}, \quad -1.$$

Synthetic division is useful for factorizing polynomials of higher degree to find the exact roots. The process to evaluate the polynomial for a right root is very tedious and time consuming. However, we can quicken the process with the tools such as rational roots test and Descartes' rule of signs, which we are not going to explore here (http://www.purplemath.com/modules/synthdiv4.html).

CHAPTER 4

CUBIC EQUATIONS

A cubic equation is a polynomial equation in which the highest occurring power of the variable is the third power. The general form may be written as:

$$\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0.$$

The coefficients α_0, ..., α_3 are real numbers. We will always assume that α_3 is non-zero (otherwise it is a quadratic or lower degree equation).

Solving a cubic equation amounts to finding the roots of a cubic function.

4.1 Cardano's Method

The first cubic equations to be solved were those of the form :

$$x^3 + px + q = 0 \tag{4.1}$$

The key to solving this equation is to note that the apparent more complicated equation:

$$w^6 + Bw^3 + C = 0$$

is in fact easy to solve because it is a quadratic in w^3:

$$(w^3)^2 + Bw^3 + C = 0.$$

We would like to convert the original equation, $x^3 + px + q = 0$, into this easily solved form by using a simple substitution. This might appear to be impossible until we realise