A STUDY ON *Q* CHART FOR SHORT RUNS

by

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KAJIAN CARTA *Q* UNTUK LARIAN PENDEK

ABSTRAK

Suatu peralatan yang penting dalam kawalan mutu ialah carta kawalan Shewhart. Kelemahan carta kawalan Shewhart adalah ia hanya dikhaskan untuk pengeluaran bervolum tinggi. Walau bagaimanapun, sejak kebelakangan ini, wujud kecenderungan di kalangan pengeluar untuk menghasilkan lot bersaiz kecil atau pengeluaran bervolum rendah. Kecenderungan ini disebabkan teknik tepat pada masa, pengeluaran serentak, penyediaan kerja kedaian, pengurangan inventori dalam proses dan nilai kos yang semakin dititikberatkan. Keadaan ini bertentangan dengan pengeluaran besar-besaran yang mana saiz lot adalah besar dan data yang sedia ada tidak menimbulkan masalah untuk memulakan proses penggunaan carta kawalan. Justeru, carta kawalan Shewhart tidak sesuai untuk digunakan dalam pengeluaran bervolum rendah. Istilah yang digunakan untuk menggambarkan pengeluaran bervolum rendah sedemikian ialah "Pengeluaran Larian Pendek" atau lebih dikenali sebagai "Larian Pendek". Dalam situasi ini, bilangan saiz sampel yang digunakan adalah kurang daripada 50. Oleh itu, masalah utama yang dihadapi adalah banyak carta kawalan diperlukan untuk mencartakan pelbagai proses yang berlainan. Hal ini akan merumitkan dan menyusahkan tugas seseorang juruinspeksi kawalan kualiti. Justeru, banyak kajian telah dijalankan untuk mengubahsuai carta kawalan yang sedia ada untuk digunakan dalam situasi larian pendek ini. Antara carta-carta kawalan larian pendek yang telah dicadangkan pada hari ini ialah carta *Q*, carta Zed, carta perbezaan dan carta sisihan daripada nominal. Objektif disertasi ini adalah untuk membincangkan pelbagai jenis carta kawalan untuk larian pendek yang sedia

ada dan untuk menjalankan suatu kajian simulasi bagi menilai prestasi purata panjang larian (ARL) carta Q untuk min proses berdasarkan ukuran-ukuran individu. Semua kajian simulasi dalam disertasi ini dijalankan dengan menggunakan program "Sistem Analisis Berstatistik (SAS)".

ABSTRACT

An important tool in quality control is the Shewhart control chart. The disadvantage of a Shewhart control chart is that it is only used for high volume manufacturing. However, in recent years, there exist a trend among manufacturers to produce smaller lot sizes or low volume manufacturing. This trend is due to just-in-time techniques (JIT), synchronous productions, job-shop settings and the reduction of in-process inventories and costs. This situation contradicts with high volume production where the lot size is large and initializing a control charting process is not a problem as the data is readily available. Therefore, a Shewhart control chart is not suitable for use in low volume production. The term used to describe such a low volume production is "Short Runs Production" or more commonly, "Short Runs". In this situation, the sample size is less than 50 and frequent changes from process to process exists. Thus, a major problem faced is the need to chart a large number of different processes and the consequent large number of charts required. This situation will make the work of a quality control inspector more difficult. Therefore, a lot of research have been made to modify the current control charts so that they can be applied in a short runs environment. To date, several short runs control charts that have been suggested are the Q charts, Zed charts, difference charts and deviation from nominal charts. The objectives of this dissertation are to review the various types of short runs control charts that are available as well as to conduct a simulation study to evaluate the average run length (ARL) performance of the Q chart for the process mean based on individual measurements. All the simulation

studies in this dissertation are made using the "Statistical Analysis System (SAS)" program.

CHAPTER 1

INTRODUCTION

1.1 AN INTRODUCTION ON SHORT PRODUCTION RUNS

Short production runs involves processes that produce products to fulfil customers' specific needs and requirements. Now, short production runs is a necessity in most manufacturing industries and it will become more important in the future.

Short runs involve low-volume manufacturing in smaller lot sizes. This is due to just-in-time techniques, job-shop settings, inventory control in a process and built to order production.

The main problem faced in low-volume manufacturing is to estimate the process parameters with the limited data available and to compute the control limits before the start of a production run. Thus, many control charts are needed due to the existence of many different processes. Two main requirements in short runs are as follows (Quesenberry, 1991) :

i) The use of the first few units of production to compute the process parameters.

ii) Plotting of all statistics on a standard scale so that the different variables in a process can all be plotted on the same control chart to simplify the chart's management program.

Thus, traditional statistical quality control methods should be modified so that they can be used in short runs.

To date, numerous works on the use of short runs control charting procedures have been suggested. Bothe (1989) and Burr (1989) suggested using target specification values for process parameters to construct charts. However, the choice of control limits based on this method produce higher false alarm rates. To solve this problem, Quesenberry (1991) introduced Q charts for attribute and variable data. Basically, the Q charts based on attribute data are the binomial, Poisson and geometric Q charts. Q charts for variables data are based on individual measurements and subgrouped data. The Q charts for variables data are used to control the process mean and process variance. Using some transformation methods, the Q statistics are all independent and identically distributed standard normal random variables. This transformation maintains the information of the original statistics but all the Q statistics can be plotted on a standard scale.

Wheeler (1991) suggested a few short runs variable control charts. They are the Zed chart, Z^* chart, difference chart and Zed-bar chart.

1.2 RESEARCH OBJECTIVE

The research objectives are to review the various types of short runs control chart and to compare the performances of the Q charts for the process mean based on individual measurements using several tests or runs rules. Four different cases of the process mean and variance when they are known and unknown are considered.

1.3 ORGANIZATION OF THE DISSERTATION

This section discusses a summary of the organization of this dissertation. In Chapter 1, we introduce the idea of short production runs. The problems of short runs and the solutions to such problems are discussed. Chapter 1 also describes the objective of this research.

Chapter 2 discusses the basic concepts that will be used in the subsequent chapters. These include the normal distribution, average run length (ARL) and Shewhart control charts, which include the $\overline{X}-R$ and $\overline{X}-S$ charts.

In Chapter 3, the Q chart for variable data, Q chart for attribute data and Wheeler's method will be discussed. The Q chart for variable data monitors a shift in the mean of individual measurements or subgrouped data. The Q chart for attribute data involves either the binomial Q chart, geometric Q chart or Poisson Q chart.

Chapter 4 is the most important chapter in this dissertation. It discusses the performance of the Q chart in controlling the process mean of individual measurements. The simulation studies in this chapter are conducted using the "*Statistical Analysis System* (SAS)" program.

Chapter 5 summarizes the main research in this dissertation and discusses future research that can be made in short production runs.

CHAPTER 2

SOME PRELIMINARIES AND BASIC CONCEPTS

2.1 UNIVARIATE NORMAL DISTRIBUTION

The normal distribution deals with a mathematical function that is used to describe the random behaviour of a measurable characteristic in a population or process (Pitt, 1993). The probability density function of a random variable, X which follows a normal distribution, is (Montgomery, 2009)

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},-\infty<x<\infty \tag{2.1}$$

The normal distribution is an important distribution in quality control and it is a theoretical basis in constructing control charts. The parameters of a normal distribution are the mean, μ $(-\infty<\mu<\infty)$ and variance, $\sigma^2\,(>0)$.

We always use $X\sim N(\mu,\sigma^2)$ to show that X is a random variable having a normal distribution with mean μ and variance σ^2. The normal distribution is symmetry and has a bell shape. Figure 2.1 shows a normal distribution (Pitt, 1993).

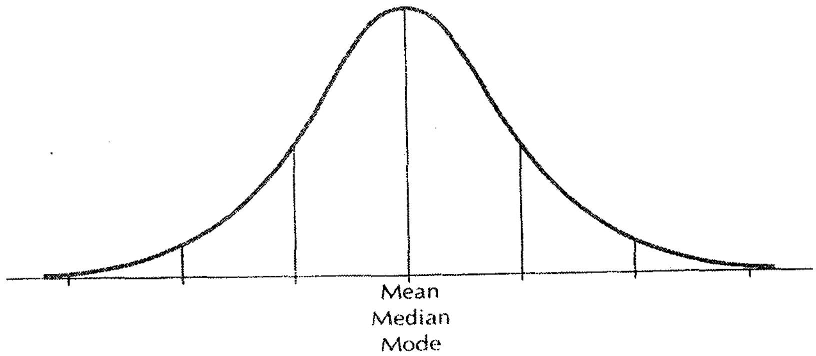

Figure 2.1. A normal distribution

Another property of a normal distribution that should be considered is the curvature. The curvature that goes down is a convex and at some point the curvature changes to a concave. The point that the curvature changes is called the point of inflection.

Under the normal curve, 68.26% is an area between $\mu \pm 1\sigma$, 95.44% is an area between $\mu \pm 2\sigma$ and 99.73% is an area between $\mu \pm 3\sigma$. The balance 0.27% is an area outside of $\mu \pm 3\sigma$. These percentages are summarized in Figure 2.2 (Pitt, 1993).

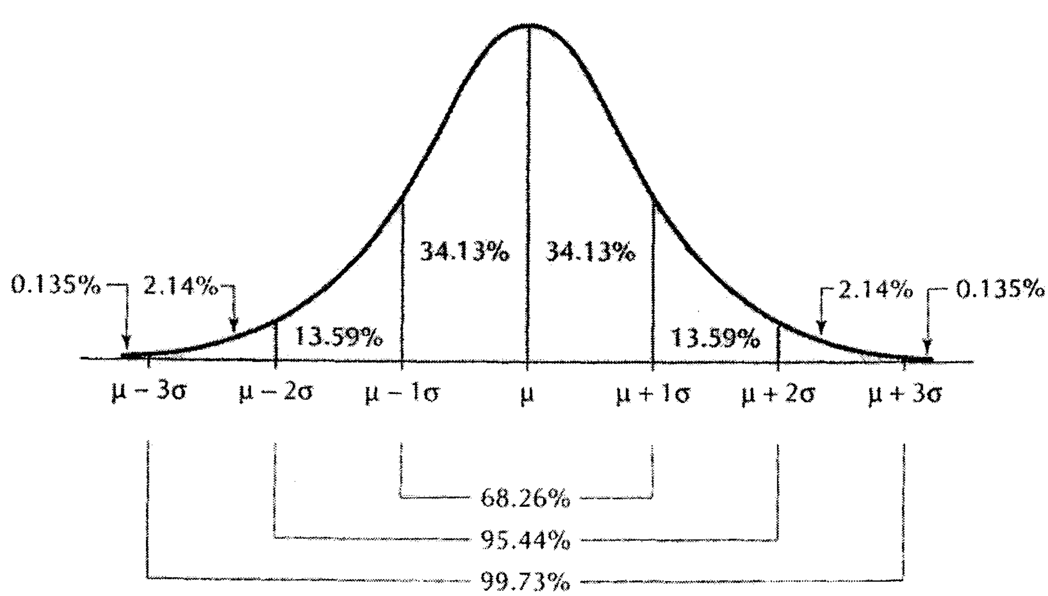

Figure 2.2: A normal curve accounting for 100% of the total area

The cumulative distribution function of a normal random variable, X is defined as the probability that X less than or equal to a value, say, a (see Equation 2.2) (Montgomery, 2009).

$$P(X \leq a) = F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (2.2)$$

By using the transformation,

$$Z = \frac{X-\mu}{\sigma}, \qquad (2.3)$$

the definite integral in Equation (2.2) can be solved. Here, Equation (2.2) reduces to (Montgomery, 2009)

$$P(X \leq a) = P\left(Z \leq \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right), \qquad (2.4)$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. The transformation in Equation (2.3) transforms the random variable $X \sim N(\mu, \sigma^2)$ into the standard normal random variable $Z \sim N(0,1)$.

2.2 AVERAGE RUN LENGTH

Average run length (ARL) is the average number of points that must be plotted on a control chart before the chart indicates the first out-of-control signal (Montgomery, 2009). It is one of the measures of a control chart's performance. If the process observations are uncorrelated, which means independence among the observations, then the number of points is a geometric random variable with parameter p. The mean of a geometric distribution is $\frac{1}{p}$. Thus, the ARL can be calculated as (Montgomery, 2009)

$$ARL = \frac{1}{p} . \tag{2.5}$$

Note that p denotes the probability of an arbitrary point plotting beyond the control limits.

For the in-control ARL (ARL_0), (Montgomery, 2009)

$$ARL_0 = \frac{1}{\alpha} \tag{2.6}$$

and for the out-of-control ARL (ARL_1),

$$ARL_1 = \frac{1}{1-\beta}, \tag{2.7}$$

where α and β denote the Type-I and Type-II errors, respectively.

2.3 SHEWHART $\overline{X}$ CONTROL CHARTS

Shewhart control charts were developed by Dr. Walter A.Shewhart, a member of the technical staff at Bell Telephone Laboratories in New York. A control chart is the most original and remarkable statistical tool in statistical process control (SPC). They allow users to make decisions based on samples from a process. But control charts were not fully accepted since they were developed. Many control charting programs degenerated into record-keeping activities, an ineffective use of a powerful quality control tool.

In the beginning of 1950s, control charts have the most influence on statistical quality control methods (Montgomery, 2009). Dr. W. Edwards Deming, a practising statistician in Washington D.C, was identified as the individual who deserved credit to these changes. He taught statistical quality control methods to Japanese engineers and managers in the 1950s (Pitt, 1993).

Control charts are used to monitor manufacturing processes to identify the presence of special causes that are responsible for a change in the process that resulted in excessive variation. The special causes responsible for a shift in the process should be determined so that corrective actions can be taken. If a process adjustment is made without searching for the special causes, the causes may still be present and excessive variation may occur again in the process.

Control charts are used to check whether a process is in-control or out-of-control. A process is said to be in statistical control if it behaves in a random, stable and predictable manner. When certain problems in a process occur that cause the process variability to increase beyond the level attributed to chance causes, the process is said to be out-of-control.

2.3.1 CONTROL CHARTS FOR $\overline{X}$ AND *R*

The $\overline{X}$ control chart is used to monitor the process mean while the *R* control chart is used to monitor the process variance. The $\overline{X}$ and *R* charts are among the most important and useful online statistical process monitoring and control techniques (Montgomery, 2009).

Assume that a process is normally distributed with mean μ and standard deviation σ. If $X_1, X_2, ..., X_n$ is a sample of size *n*, then

$$\overline{X} = \frac{X_1 + X_2 + ... + X_n}{n} \tag{2.8}$$

is the average of this sample. Here, $\overline{X}$ is also normally distributed with mean μ and standard deviation $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$. The probability is $1-\alpha$ that any sample mean will fall between (Montgomery, 2009)

$$\mu - Z_{\alpha/2}\sigma_{\overline{X}} \text{ and } \mu + Z_{\alpha/2}\sigma_{\overline{X}}. \tag{2.9}$$

The value of $Z_{\alpha/2}$ will be replaced by 3 to maintain the three-sigma limits. Equation (2.9) is used on a control chart for the sample means as the upper and lower control limits. These control limits are only valid when μ and σ are both known. If a sample mean falls beyond these limits, it shows that the process mean is not equal to μ.

In reality, usually we will not know the values of μ and σ. Therefore, both μ and σ have to be estimated from preliminary samples assumed to be in-control. Suppose that m preliminary samples are available and each sample contains n observations. Usually m is at least 20 or 25 and n is small, either 4, 5 or 6. Let $\overline{X}_1, \overline{X}_2, ..., \overline{X}_m$ be the average of these samples. Then, the best estimator of μ is

$$\overline{\overline{X}} = \frac{\overline{X}_1 + \overline{X}_2 + ... + \overline{X}_m}{m}, \tag{2.10}$$

where $\overline{\overline{X}}$ is used as the center line on the $\overline{X}$ chart. To estimate the standard deviation, σ, we may use the ranges of the m samples. If $X_1, X_2, ..., X_n$ is a sample of size n, then the range of this sample is

$$R = X_{\max} - X_{\min}. \tag{2.11}$$

Here, $X_{\max}$ and $X_{\min}$ represent the largest and smallest observations in the sample. If we let $R_1, R_2, ..., R_m$ be the ranges of the m samples, the average range is

$$\overline{R} = \frac{R_1 + R_2 + ... + R_m}{m}. \tag{2.12}$$

Let $W = \frac{R}{\sigma}$ be a random variable. Then W is called the relative range. The estimator of σ is (Montgomery, 2009)

$$\hat{\sigma} = \frac{\overline{R}}{d_2}, \tag{2.13}$$

where d_2 is the mean of W.

Therefore, the control limits of the $\overline{X}$ chart are (Montgomery, 2009)

$$UCL = \overline{\overline{X}} + A_2\overline{R}$$

$$\text{Center line} = \overline{\overline{X}} \tag{2.14}$$

$$LCL = \overline{\overline{X}} - A_2\overline{R},$$

where $$A_2 = \frac{3}{d_2\sqrt{n}}.$$

To determine the control limits of the R chart, we need to estimate σ_R. By assuming that the quality characteristic is normally distributed, $\hat{\sigma}_R$ could be found from the distribution of the relative range $W = \frac{R}{\sigma}$. The standard deviation is d_3 which is a function of n. The standard deviation of R is (Montgomery, 2009)

$$\sigma_R = d_3\sigma \tag{2.15}$$

since $R = W\sigma$. If σ is unknown, we could estimate σ_R as follows:

$$\hat{\sigma}_R = d_3\frac{\overline{R}}{d_2}. \tag{2.16}$$

Therefore, the limits of the R chart with the usual three-sigma control limits are (Montgomery, 2009)

$$UCL = \overline{R} + 3\hat{\sigma}_R = \overline{R} + 3d_3\frac{\overline{R}}{d_2}$$

$$\text{Center line} = \overline{R} \tag{2.17}$$

$$LCL = \overline{R} - 3\hat{\sigma}_R = \overline{R} - 3d_3\frac{\overline{R}}{d_2}$$

Define $D_4 = 1 + 3\frac{d_3}{d_2}$ and $D_3 = 1 - 3\frac{d_3}{d_2}$, then Equation (2.17) reduces to

$$UCL = \bar{R}D_4$$

$$\text{Center line} = \bar{R} \tag{2.18}$$

$$LCL = \bar{R}D_3$$

2.3.2 CONTROL CHARTS FOR $\bar{X}$ AND S

The sample standard deviation, S could also be used to estimate the standard deviation, σ. Generally, the $\bar{X}$ and S charts are preferable to their more familiar counterparts, the $\bar{X}$ and R charts when the sample size, n is moderately large, say $n > 10$ (Montgomery, 2009).

The procedure used to set up the $\bar{X}$ and S charts are about the same as that for the $\bar{X}$ and R charts. The difference is that we must compute the sample standard deviation, S, instead of the sample range, R. Note that (Montgomery, 2009)

$$S^2 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1} \tag{2.19}$$

is an unbiased estimator of σ^2. However, S is not an unbiased estimator of σ. The value of S estimates $c_4\sigma$, where c_4 is a constant that depends only on the sample size, n. Consider the case when a standard value is given for σ. From the above information, $E(S) = c_4\sigma$. Therefore, the center line for the chart is $c_4\sigma$ and the three-sigma control limits are (Montgomery, 2009)

$$UCL = c_4\sigma + 3\sigma\sqrt{1-c_4^{\ 2}}$$

and (2.20)

$$LCL = c_4\sigma - 3\sigma\sqrt{1-c_4^{\ 2}}.$$

Consequently, the limits of the S chart when parameters are known are

$$UCL = B_4\sigma$$

$$\text{Center line} = c_4\sigma \qquad (2.21)$$

$$LCL = B_3\sigma,$$

where $B_4 = c_4 + 3\sqrt{1-c_4^{\ 2}}$ and $B_3 = c_4 - 3\sqrt{1-c_4^{\ 2}}$.

The limits of the $\overline{X}$ chart when parameters are known are (Montgomery, 2009)

$$UCL = \mu + 3\frac{\sigma}{\sqrt{n}}$$

$$\text{Center line} = \mu \qquad (2.22)$$

$$LCL = \mu - 3\frac{\sigma}{\sqrt{n}}.$$

When no standard value is given for σ, analyze the past data to estimate the value of σ. Consider m preliminary samples, each of size n and let S_i be the standard deviation of the ith sample. The average of the m standard deviations is

$$\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_i. \qquad (2.23)$$

The limits of the S chart are (Montgomery, 2009)

$$UCL = B_6 \bar{S}$$

$$\text{Center line} = \bar{S} \tag{2.24}$$

$$LCL = B_5 \bar{S},$$

where $B_5 = 1 - \frac{3}{c_4}\sqrt{1-c_4^2}$, $B_6 = 1 + \frac{3}{c_4}\sqrt{1-c_4^2}$ and $\frac{\bar{S}}{c_4}$ is an unbiased estimator of σ. Also, we can see that $B_6 = \frac{B_4}{c_4}$ and $B_5 = \frac{B_3}{c_4}$. By knowing that $\bar{\bar{X}}$ is an estimator for μ and $\frac{\bar{S}}{c_4}$ is an estimator for σ, the $\bar{X}$ chart's limits are

$$UCL = \bar{\bar{X}} + A_3 \bar{S}$$

$$\text{Center line} = \bar{\bar{X}} \tag{2.25}$$

$$LCL = \bar{\bar{X}} - A_3 \bar{S},$$

where $A_3 = \frac{3}{c_4\sqrt{n}}$.

CHAPTER 3

A REVIEW ON SHORT RUNS CONTROL CHARTS

3.1 Q CHARTS

3.1.1 Q CHARTS FOR VARIABLE DATA

All the Q charts that have been proposed by Quesenberry (1995a) for controlling the process mean and variance based on individual measurements and subgrouped data will be discussed in this section. The Q statistics for each case are identically and independently distributed standard normal random variables. The 1-of-1, 9-of-9, 3-of-3 and 4-of-5 tests are used to identify the existence of parameter shifts in a production process. The notations in Table 3.1 below will be used for certain distribution functions.

Table 3.1. Notations for Distribution Functions

$\Phi(\cdot)$	-The standard normal distribution function
$\Phi^{-1}(\cdot)$	-The inverse standard normal distribution function
$G_v(\cdot)$	-The student-t distribution function with v degrees of freedom
$H_v(\cdot)$	-The chi squared distribution function with v degrees of freedom
$F_{v_1,v_2}(\cdot)$	-The F distribution function with (v_1, v_2) degrees of freedom

Source : Quesenberry (1995a)

3.1.1.1 *Q* CHARTS BASED ON INDIVIDUAL MEASUREMENTS

Let $X_1, X_2, \ldots$ represent individual measurements taken from a production process and assume that these values are independently and identically distributed normal $N\left(\mu, \sigma^2\right)$ random variables. The sample mean and variance are defined as follows:

$$\bar{X}_r = \frac{1}{r}\sum_{j=1}^{r} X_j \text{ , } r = 1,2,\ldots \tag{3.1}$$

$$S_r^2 = \frac{1}{r-1}\sum_{j=1}^{r}(X_j - \overline{X}_r)^2 \text{ , } r = 2,3,\ldots \tag{3.2}$$

By Young and Cramer (1971), these formulae can also be written as follows:

$$\bar{X}_r = \frac{1}{r}\left[(r-1)\bar{X}_{r-1} + X_r\right], r = 2,3,\ldots \tag{3.3}$$

$$S_r^2 = \frac{r-2}{r-1}S_{r-1}^2 + \frac{1}{r}\left(X_r - \bar{X}_{r-1}\right)^2, r = 3,4\ldots \tag{3.4}$$

3.1.1.1.1 *Q* STATISTICS FOR THE PROCESS MEAN, μ

In order to compute the Q statistics for monitoring the process mean, individual measurements will be transformed into four different cases as follows (Quesenberry, 1995a):

Case KK: $\mu = \mu_0, \sigma = \sigma_0$, both known

$$Q_r(X_r) = \frac{X_r - \mu_0}{\sigma_0} \text{ , } r = 1,2,\ldots \tag{3.5}$$

Case UK: μ unknown, $\sigma = \sigma_0$ known

$$Q_r(X_r) = \left(\frac{r-1}{r}\right)\left(\frac{X_r - \bar{X}_{r-1}}{\sigma_0}\right), \; r = 2,3,... \tag{3.6}$$

Case KU: $\mu = \mu_0$ known, σ unknown

$$Q_r(X_r) = \Phi^{-1}\left\{G_{r-1}\left(\frac{X_r - \mu_0}{S_{0,r-1}}\right)\right\}, \; r = 2,3,..., \tag{3.7}$$

where

$$S_{0,r-1} = \frac{1}{r}\sum_{j=1}^{r}\left(X_j - \mu_0\right)^2 . \tag{3.8}$$

Case UU: μ and σ both unknown

$$Q_r(X_r) = \Phi^{-1}\left\{G_{r-2}\left[\left(\frac{r-1}{r}\right)^{\frac{1}{2}}\left(\frac{X_r - \bar{X}_{r-1}}{S_{r-1}}\right)\right]\right\}, \; r = 3,4,... \tag{3.9}$$

The statistic for Case KK is similar to that of the conventional control chart since the values of mean and variance are both known. The first value of the Q statistic for cases UK and KU cannot be obtained. Similarly, the first and second values of the Q statistics for Case UU cannot be computed. This is due to the existence of unknown parameters for these three cases. The result of Case UU is very important since it can be used to control the mean of a normal distributed process at the beginning of a process, where both the mean and standard deviation are unknown. Case UU solves the short runs control charting problem.

3.1.1.1.2 Q STATISTICS FOR THE PROCESS VARIANCE, σ^2

Consider the following two different cases. For these 2 cases, let (Quesenberry, 1995a)

$$R_r = X_r - X_{r-1} \tag{3.10}$$

Case K: μ unknown and $\sigma = \sigma_0$ known

$$Q(R_r) = \Phi^{-1}\left\{H_1\left(\frac{R_r^2}{2\sigma_0^2}\right)\right\},\ r = 2,4... \tag{3.11}$$

Case U: μ and σ both unknown

$$Q(R_r) = \Phi^{-1}\left\{F_{1,v}\left(\frac{vR_r^2}{R_2^2 + R_4^2 + ... + R_{r-2}^2}\right)\right\},$$

$$r = 4, 6, ...; v = \frac{r}{2} - 1 \tag{3.12}$$

Quesenberry (1991) showed that the Q statistics in Equations (3.5), (3.6), (3.7), (3.9), (3.11) and (3.12) are identically and independently distributed standard normal random variables. These statistics can be plotted on the Shewhart control chart with control limits at ± 3.

3.1.1.2 Q CHARTS FOR SUBGROUPED DATA

It will sometimes be useful to form Q charts from the sample means, $\overline{X}_i$ and sample variance, S_i^2 for data grouped into samples of size n_i as in the below table.

Table 3.2. Sample and Statistics Notations

Sample	Sample Mean	Sample Variance
X_{11} X_{12} $\cdots$ X_{1n_1}	$\bar{X}_1$	S_1^2
X_{21} X_{22} $\cdots$ X_{2n_2}	$\bar{X}_2$	S_2^2
$\vdots$ $\vdots$ $\vdots$	$\vdots$	$\vdots$
X_{r1} X_{r2} $\cdots$ X_{rn_r}	$\bar{X}_r$	S_r^2

Source : Quesenberry (1995a)

Let

$$\bar{\bar{X}}_r = \frac{n_1\bar{X}_1 + n_2\bar{X}_2 + \ldots + n_r\bar{X}_r}{n_1 + n_2 + \ldots + n_r} \tag{3.13}$$

and

$$S_{p,r}^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2 + \ldots + (n_r-1)S_r^2}{n_1 + n_2 + \ldots + n_r - r}. \tag{3.14}$$

Note that the grand sample mean, $\bar{\bar{X}}_r$ and pooled sample variance, $S_{p,r}^2$ are defined differently from $\bar{X}_r$ and S_r^2 for individual measurements (see Equations (3.1) and (3.2)). It is also necessary to state that the subscript "p" in $S_{p,r}^2$ (see Equation (3.14)) is used to differentiate $S_{p,r}^2$ from $S_{0,r}^2$ in Equation (3.18).

3.1.1.2.1 *Q* STATISTICS FOR THE PROCESS MEAN, μ

Similar to the case of individual measurements, the transformation of the sample means to produce standard normal Q statistics for controlling the process mean can be divided into four cases (Quesenberry, 1995a).

Case KK: $\mu = \mu_0, \sigma = \sigma_0$, both known

$$Q_r\left(\bar{X}_r\right)=\frac{\sqrt{n_r}\left(\bar{X}_r-\mu_0\right)}{\sigma_0},\ r=1,2,\ldots \tag{3.15}$$

Case UK: μ unknown, $\sigma=\sigma_0$ known

$$Q_r\left(\bar{X}_r\right)=\sqrt{\frac{n_r\left(n_1+n_2+\ldots+n_{r-1}\right)}{n_1+n_2+\ldots+n_r}}\left(\frac{\bar{X}_r-\bar{\bar{X}}_{r-1}}{\sigma_0}\right),\ r=2,3,\ldots \tag{3.16}$$

Case KU: $\mu=\mu_0$ known and σ unknown

$$Q_r\left(\bar{X}_r\right)=\Phi^{-1}\left\{G_{n_1+\ldots+n_r}\left(\frac{\sqrt{n_r}\left(\bar{X}_r-\mu_0\right)}{S_{0,r}}\right)\right\},\ r=2,3,\ldots, \tag{3.17}$$

where

$$S_{0,r}^2=\frac{\sum_{\alpha=1}^{r}\sum_{j=1}^{n_\alpha}\left(X_{\alpha j}-\mu_0\right)^2}{n_1+\ldots+n_r}. \tag{3.18}$$

Case UU: μ and σ both unknown

$$Q_r\left(\bar{X}_r\right)=\Phi^{-1}\left\{G_{n_1+\ldots+n_r-r}\left(\sqrt{\frac{n_r\left(n_1+\ldots+n_{r-1}\right)}{n_1+\ldots+n_r}}\left(\frac{\left(\bar{X}_r-\bar{\bar{X}}_{r-1}\right)}{S_{p,r}}\right)\right)\right\},$$

$$r=2,3,\ldots \tag{3.19}$$

The Q statistics in Equations (3.15), (3.16), (3.17) and (3.19) are independently and identically distributed standard normal, $N(0,1)$ random variables under the stable or in control normality assumption and are approximately so for many other process distributions (Quesenberry,1995a). The Q statistics in Equation (3.15) is to test that the mean of the rth sample has a particular value $\mu=\mu_0$ when $\sigma=\sigma_0$ is known.

The Q statistics are appropriate test statistics for particular hypothesis testing problems. For example, when $r = 2$ and n is a constant, the argument in Equation (3.19) is a Student-t statistic with $2n-2$ degrees of freedom to test that the first and second samples are from distributions with the same mean, assuming that they are normally distributed with the same unknown variance. For all values of r, this Student-t statistic has $(n-1)r$ degrees of freedom. The test and chart's performance based on Equation (3.19) will approach the Q statistic in Equation (3.15) that assumes the parameters are known as r increases.

3.1.1.2.2 Q STATISTICS FOR THE PROCESS VARIANCE, σ^2

In this section, we will consider the Q statistics for controlling the process variance, σ^2. Two different cases, i.e., σ known and unknown are considered. The data used are subgrouped data. The following Q statistics are obtained from Quesenberry (1995a).

Case K: $\sigma = \sigma_0$, σ^2 known

$$Q_r\left(S_r^2\right) = \Phi^{-1}\left\{H_{n_r-1}\left(\frac{(n_r-1)S_r^2}{\sigma_0^{\;2}}\right)\right\},\; r = 1,2,\ldots \qquad (3.20)$$

Case U: σ^2 unknown

Let

$$w_r = \frac{(n_1+\ldots+n_{r-1}-r+1)S_r^2}{(n_1-1)S_1^2+\ldots+(n_{r-1}-1)S_{r-1}^2} \qquad (3.21)$$

and

$$Q_r\left(S_r^2\right)=\Phi^{-1}\left\{F_{n_r-1,n_1+\ldots+n_{r-1}-r+1}\left(w_r\right)\right\},\ r=2,3,\ldots \tag{3.22}$$

Under the assumption that the underlying process follows a $N\left(\mu,\sigma^2\right)$ distribution, then the Q statistics in Equations (3.20) and (3.22) are sequences of independent normal, $N(0,1)$ random variables. The argument in Equation (3.20), i.e., $\frac{\left(n_r-1\right)S_r^2}{\sigma_0^2}$ is a chi-squared distributed statistic to test that the sample variance, S_r^2 is from a normal distribution with variance, ${\sigma_0}^2$. The statistic w_r in Equation (3.21) is a snedecor F variable, with n_r-1 and $n_1+\ldots+n_{r-1}-r+1$ degrees of freedom for testing that the *r*th sample is from a distribution with the same variance as the $r-1$ preceding samples. As r increases, the power of the hypothesis test and the performance of the corresponding chart for case U will both approach that of case K.

3.1.1.3 TESTS FOR PARAMETER SHIFTS

There are many possible tests that can be made on a Shewhart Q chart to detect a shift in μ or σ for a normal distributed process. In this section, we will discuss four tests. Three of them, i.e., the 1-of-1 test, 9-of-9 test and 4-of-5 test were given by Nelson (1984). Quesenberry (1995a) suggested that the Q statistics be plotted on Shewhart charts with control limits at ± 3. For illustration, an increase in the mean will be represented by "A" and a decrease by "B".

Test 1:

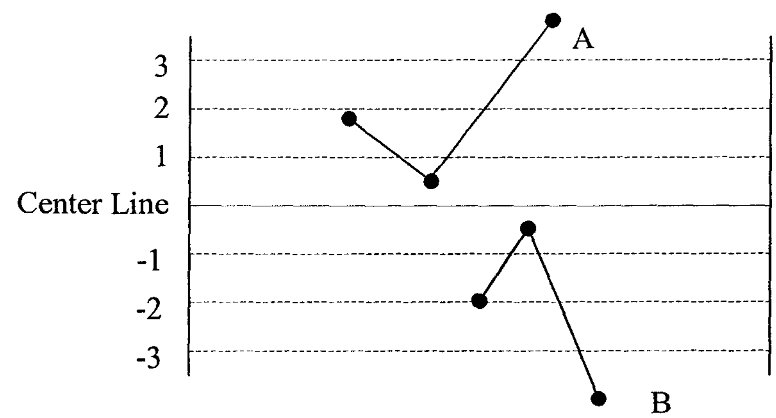

Figure 3.1. The 1-of-1 test

Source : Nelson (1984)

This test shows an increase in mean, μ if one point is more than 3 and a decrease in mean, μ if one point is less than -3. This is the classical Shewhart's test.

Test 2:

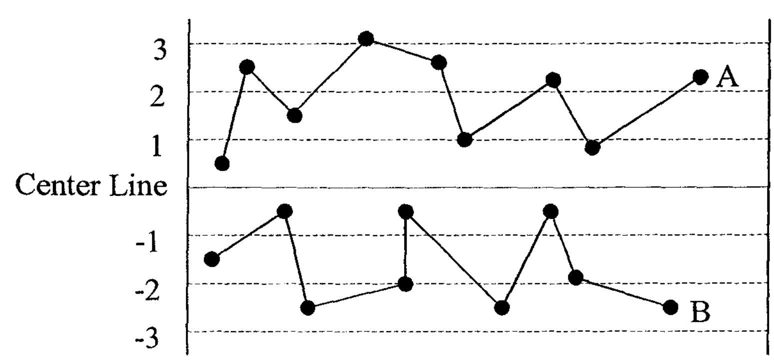

Figure 3.2. The 9-of-9 test

Source : Nelson (1984)

This test shows an increase in mean, μ when all nine consecutive points are plotted above the center line and a decrease in mean, μ when all nine consecutive points are plotted below the center line. This test is only possible when nine consecutive points are available.