THE CONDITION OF QUADRATIC AND CUBIC BEZIER CURVES TO TOUCH A CONSTRAINT LINE

by

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SYARAT BAGI LENGKUNG BEZIER KUADRATIK DAN KUBIK UNTUK MENYENTUH GARIS KEKANGAN

ABSTRAK

Bezier merupakan satu daripada polinomial yang paling berpengaruh untuk interpolasi. Interpolasi lengkung Bezier selalu berada di dalam hull cembung titik kawalan dan tidak pernah bergerak terlampau jauh dari titik kawalan. Interpolasi polinomial Bezier mempunyai aplikasi yang sangat luas kerana ia senang untuk dikira dan sangat stabil.

Di dalam disertasi ini, kami menyelidik syarat lengkung Bezier kuadratik dan kubik untuk menyentuh garis kekangan. Jadi, titik kawalan memainkan peranan penting untuk mencapai matlamat ini kerana titik kawalan boleh bergerak ke seluruh arah dan memberikan pelbagai bentuk lengkung Bezier. Walaubagaimanapun, titik kawalan pertama dan terakhir akan diberikan dalam disertasi ini. Oleh itu, hanya titik kawalan di tengah sahaja yang akan mengubah bentuk bagi setiap lengkung Bezier ini. Oleh hal demikian, bagi menentukan syarat lengkung Bezier kuadratik dan kubik untuk menyentuh garis kekangan, kami perlu mencari lokasi setiap titik kawalan yang berada di tengah lengkung Bezier supaya lengkung Bezier hanya menyentuh tetapi tidak melepasi garis kekangan.

ABSTRACT

Bezier is one of the most influential polynomial representations for interpolation. The Bezier interpolating curve always lies within the convex hull of its control points and it never oscillates wildly away from the control points. Bezier polynomial interpolation has wide applications because it is easy to compute and is also very stable.

In this dissertation, discussion is made on the conditions for quadratic and cubic Bezier curve to touch a constraint line. Thus, control points play important role in order to achieve this goal because control points can oscillate and will illustrate a various shape of Bezier curve. However, the first control point and the last control point of quadratic and cubic Bezier curve will be given in this dissertation. Hence, only the middle control point will change the whole shape of each Bezier curve. Therefore, in order to determine the condition of quadratic and cubic Bezier curve so as to touch a constraint line, the location of each middle control points has to be identified so that the curve will only touch the constraint line but without crossing it.

CHAPTER 1

INTRODUCTION

1.1 Background

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics concerned with algorithms for the design of smooth curves and surfaces and for their efficient mathematical representations. The representations are used for the computation of curves and surfaces, as well as geometrical quantities of importance such as curvatures, intersections curves between two surfaces and offset surfaces (Joy, 2000).

The Bezier curve is an important part of almost every computer graphics illustration program and Computer Aided Geometric Design (CAGD) system in use today. It is used in many ways, from designing the curves and surfaces of automobiles until to defining the shape of letters in type fonts. Since it is numerically and the most stable of all the polynomial-based curves used in these applications, the Bezier curve is an ideal standard for representing the more complex piecewise polynomial curve (Mortenson, 1999). Besides, in vector graphics, a Bézier curve is an important tool used to model smooth curves that can be scaled indefinitely. Bézier curves are also used in animation as a tool to control motion (Wikipedia, 2010).

The Bezier curves were first developed independently by two Frenchmen, Paul de Casteljau and Pierre Bezier at two car companies, Renault and Citroen, during the period 1958-1960 (Farin & Hansford, 2000). In 1970, R.Forrest discovered the connection between the Bezier's work and Bernstein polynomials. The underlying mathematical theory of Bezier's work is based on the concept of Bernstein polynomials.

1.2 Statement of Problem

In this dissertation, a discussion is made on how to generate a Bezier curve so that it touches a given constraint line. However, there are some conditions that need to be considered in order to generate the curve. So, these conditions have to be determined so that a Bezier curve will touch any given constraint line.

1.3 Literature Review

In the case of constrained interpolation, Goodman et al. (1991) had proposed a scheme for the construction of a G^2 parametric interpolating curve which lies on the same side of a given set of constraint lines as the data. The interpolant is actually a parametric rational cubic Bezier curve. Meek et al. (2003), extended this result to generate the curve for a given set of ordered planar points lying on one side of a polyline, a planar G^2 interpolating curve to these data which lies on the same side of the polyline as data. After that, Jeok and Ong (2006) generated G^1 curves which are constrained to lie on the same side of the given constraint lines as the data and the interpolant is actually a parametric cubic Bezier-like curve. They also did a

modification so that the curve from across the given line would just touch the given line.

1.4 Objectives of Dissertation

The objectives of this dissertation are:

- to determine the condition of quadratic Bezier curve so that it can touch a constraint line.
- to determine the condition of cubic Bezier curve so that it can touch a constraint line.
- to show the application of Bezier curve with a constraint line.

1.5 Scope of Study

This study will investigate the condition of quadratic and cubic Bezier curve in order to touch a constraint line where the constraint line is x-axis and any straight line where the tangent slope of the straight line is positive or $m \geq 0$. In addition, the first control point and the last control point will be of positive values and they will be provided.

1.6 Significance of Study

This study is useful for those who are interested to do animation films or to make advertisements in animation.

1.7 Summary and Structure of Dissertation

This dissertation consists of six chapters and the summary of the major contribution of this dissertation is as follows:

Chapter 1 presents a brief account of Computer Aided Geometry Design (CAGD) and the background of Bezier Curve. The objectives of the dissertation, statement of problem, literature review, scope of study and the significance of this dissertation are also included in this chapter.

Chapter 2 is a brief introduction of Bezier curve. It also explains some properties of Bernstein polynomial and Bezier curve for degree one, degree two and degree three. The graphical examples of Bernstein polynomial and Bezier curve are also presented in this chapter.

Chapter 3 focus on ways of determining the condition of quadratic Bezier curve in order to touch a constraint line. First, the focus is on ways of determining the condition of quadratic Bezier curve in order to touch a constraint line where the constraint line is the x-axis. Next, this is extended to the case where the constraint line can be of any straight line.

Chapter 4 focuses on ways of determining the condition of cubic Bezier curve in order to touch a constraint line. First, the focus is on ways of determining the condition of cubic Bezier curve in order to touch a constraint line where the

constraint line is the x-axis. This is then extended to the case where the constraint line can be of any straight line.

Chapter 5 demonstrates some applications of the Bezier curve which is a 2D drawing produced through the combination of Bezier curves of different degrees. Besides, this chapter also illustrates the applications of the Bezier curve and a constraint line which can be applied in a robotic motion and parabolic wall.

Chapter 6 is the conclusion and some recommendations of this dissertation are discussed in this chapter.

CHAPTER 2

INTRODUCTION OF BEZIER CURVE

2.1 Bernstein Polynomials

The Bernstein polynomials of degree n are defined as

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \qquad 0 \le t \le 1$$

for $i = 0, 1, \ldots, n$, where the binomial coefficients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \qquad \text{if } 0 \le i \le n$$

and for mathematical convenience, it is usually set as $B_i^n = 0$ if $i < 0$ or $i > n$.

These polynomials are quite simple to be written down. The coefficient $\binom{n}{i}$ can be obtained from Pascal's triangle where the exponents on the t term increases by one as i increases and the exponents on the $(1-t)$ term decreases by one as i increases (Joy, 2000).

2.1.1 Bernstein Polynomial of Degree 1

$$B_0^1(t) = \binom{1}{0} t^0 (1-t)^{1-0}$$

$$= 1 - t$$

$$B_1^1(t) = \binom{1}{1} t^1(1-t)^{1-1}$$

$$= t$$

and can be plotted for $0 \leq t \leq 1$ as in Figure 2.1 below

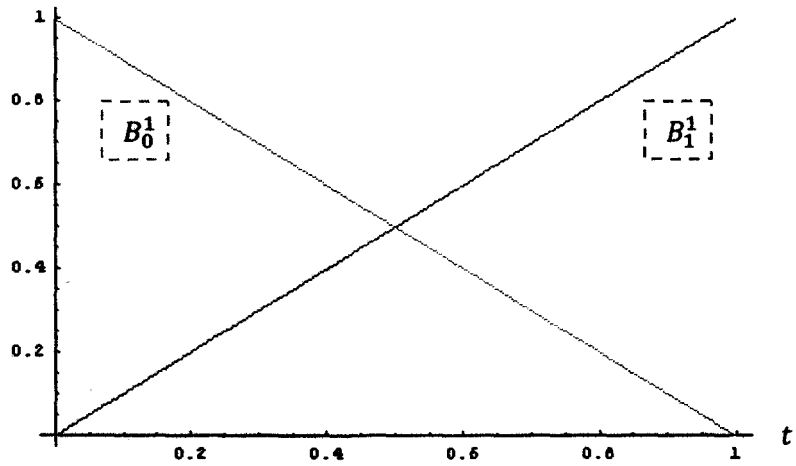

Figure 2.1: Bernstein polynomials of degree 1

2.1.2 Bernstein Polynomials of Degree 2

$$B_0^2(t) = \binom{2}{0} t^0(1-t)^{2-0}$$

$$= (1-t)^2$$

$$B_1^2(t) = \binom{2}{1} t^1(1-t)^{2-1}$$

$$= 2t(1-t)$$

$$B_2^2(t) = \binom{2}{2} t^2(1-t)^{2-2}$$

$$= t^2$$

and can be plotted for $0 \leq t \leq 1$ as in Figure 2.2 below

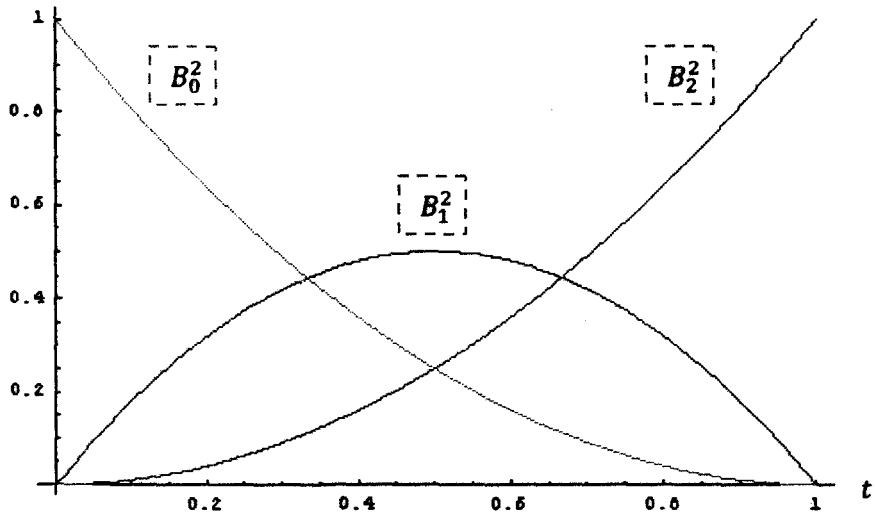

Figure 2.2: Bernstein polynomials of degree 2

2.1.3 Bernstein Polynomials of Degree 3

$$B_0^3(t) = \binom{3}{0} t^0 (1-t)^{3-0}$$

$$= (1-t)^3$$

$$B_1^3(t) = \binom{3}{1} t^1 (1-t)^{3-1}$$

$$= 3t(1-t)^2$$

$$B_2^3(t) = \binom{3}{2} t^2 (1-t)^{3-2}$$

$$= 3t^2(1-t)$$

$$B_3^3(t) = \binom{3}{3} t^3 (1-t)^{3-3}$$

$$= t^3$$

and can be plotted for $0 \leq t \leq 1$ as in Figure 2.3 below

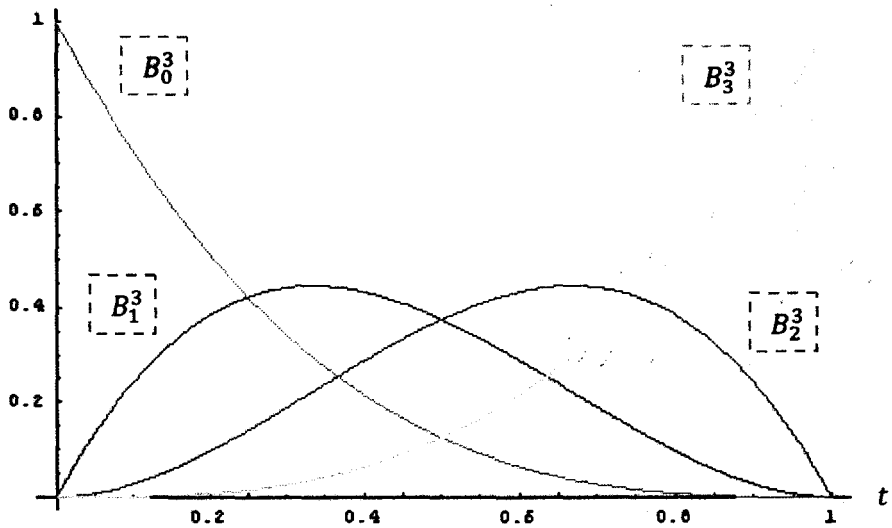

Figure 2.3: Bernstein polynomials of degree 3

2.2 Properties of Bernstein Polynomial

Bernstein basis polynomials have the following properties:

2.2.1 Endpoints

$$B_i^n(0) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$B_i^n(1) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{Otherwise} \end{cases}$$

2.2.2 Recursion

$$B_i^n = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

From right hand side,

$$(1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

$$= (1-t)\binom{n-1}{i} t^i(1-t)^{n-1-i} + t\binom{n-1}{i-1} t^{i-1}(1-t)^{n-1-(i-1)}$$

$$= \binom{n-1}{i} t^i(1-t)^{n-i} + \binom{n-1}{i-1} t^i(1-t)^{n-i}$$

$$= \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right] t^i(1-t)^{n-i}$$

$$= \binom{n}{i} t^i(1-t)^{n-i}$$

$$= B_i^n(t)$$

2.2.3 Non-Negativity

A function $f(t)$ is non-negative over an interval [a, b] if $f(t) \geq 0$ for $t \in [a, b]$. In the case of the Bernstein polynomials of degree n, each is non-negative over the interval [0,1]. In order to illustrate this, the recursion property and mathematical induction will have to be used. It can be easily seen that the functions $B_0^1(t) = 1 - t$ and $B_1^1(t) = t$ are both non-negative for $0 \leq t \leq 1$. If we assume that all Bernstein polynomials of degree less than n are non-negative, then by using the recursive definition of the Bernstein polynomial, it can be written down as

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

and argued that $B_i^n(t)$ is also non-negative for $0 \leq t \leq 1$, since all components on the right-hand side of the equation are non-negative components for $0 \leq t \leq 1$.

Through induction, all Bernstein polynomials will be non-negative for $0 \leq t \leq 1$. In this process, each of the Bernstein polynomials has been shown to be positive when $0 \leq t \leq 1$.

2.2.4 Symmetry

$$B_i^n(t) = B_{n-i}^n(1-t)$$

Proof:

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

$$= \frac{n!}{(n-(n-i))!(n-i)!} (1-(1-t))^{n-(n-i)} (1-t)^{n-i}$$

$$= B_{n-i}^n(1-t)$$

2.2.5 Derivatives

Derivatives of the nth degree Bernstein polynomials are polynomials of degree $n-1$. Based on the definition of the Bernstein polynomial, it can be shown that this derivative can be written as a linear combination of Bernstein polynomials. In particular:

$$\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) \text{ for } 0 \leq i \leq n.$$

This can be shown by direct differentiation.

$$\frac{d}{dt} B_i^n(t) = \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i}$$

$$= \binom{n}{i} \left[i t^{i-1} (1-t)^{n-i} + (n-i) t^i (1-t)^{n-i-1} (-1) \right]$$

$$= \frac{in!}{i!(n-i)!} t^{i-1}(1-t)^{n-i} - \frac{(n-i)n!}{i!(n-i)!} t^{i}(1-t)^{n-i-1}$$

$$= \frac{n(n-1)!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} - \frac{n(n-1)!}{i!(n-i-1)!} t^{i}(1-t)^{n-i-1}$$

$$= n\left(\frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!} t^{i}(1-t)^{n-i-1}\right)$$

$$= n(B_{i-1}^{n-1}(t) - B_{i}^{n-1}(t))$$

2.3 Bezier Curves

By using the Bernstein basis functions, the Bezier polynomial of degree n can be defined as:

$$P(t) = \sum_{i=0}^{n} C_i B_i^n(t) \ , \qquad 0 \leq t \leq 1$$

with coefficients $C_i \in R^r (r = 1,2,3)$.

If C_i are vectors in R^2 or R^3, then

- C_i are called as a Bezier points or control points of curve and
- $P(t)$ is a curve in the parametric form.

If C_i are real numbers R^1, then

- C_i are named as Bezier ordinates and
- $P(t)$ is a Bezier function of the variable t with Bezier points $(\frac{i}{n}, C_i)$.

The polygon $C_0C_1 \ldots C_n$ formed by connecting the Bezier points with line segments is called Bezier polygon or control polygon of the curve.

2.3.1 Linear Bezier Curve

Bezier curve of Degree 1 is written as:

$$B_0^1(t) = 1 - t$$

$$B_1^1(t) = t$$

$$P(t) = (1 - t)C_0 + tC_1$$

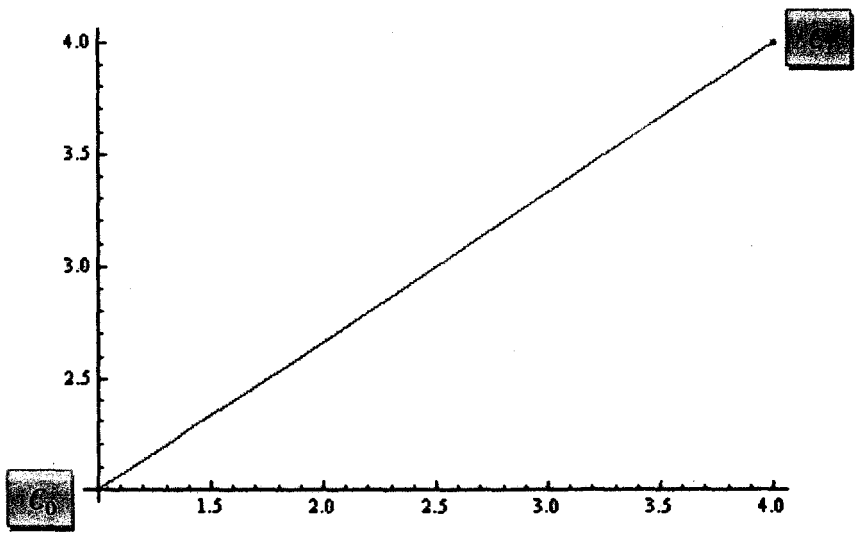

Figure 2.4: Bezier curve of degree 1

Figure 2.4 shows a straight line segment from C_0 to C_1 (a linear interpolation) and the curve lies in the convex hull of the Bezier polygon.

2.3.2 Quadratic Bezier Curve

Bezier curve of Degree two is written as:

$$B_0^2(t) = (1-t)^2$$

$$B_1^2(t) = 2t(1-t)$$

$$B_2^2(t) = t^2$$

$$P(t) = (1-t)^2 C_0 + 2t(1-t)C_1 + t^2 C_2$$

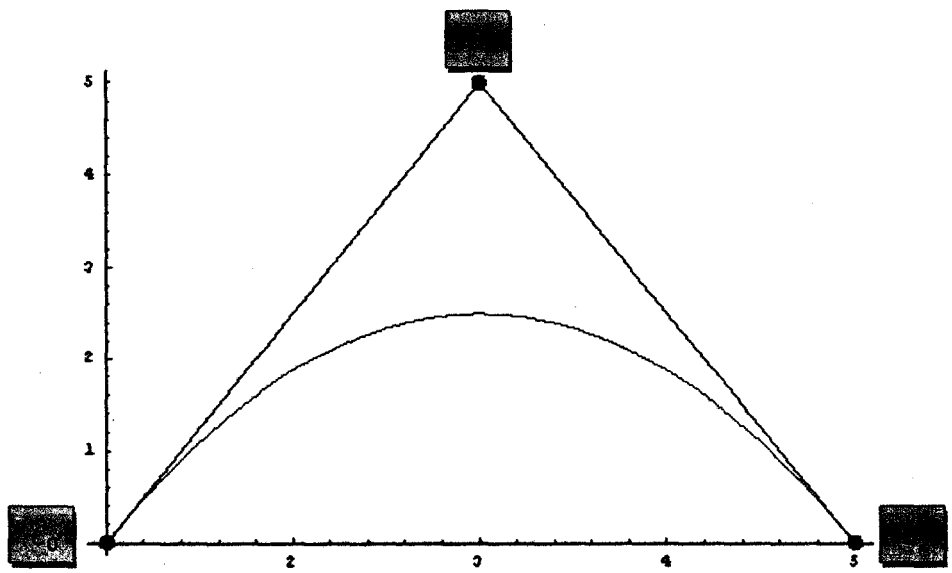

Figure 2.5: Bezier curve of degree 2

Figure 2.5 shows a parabolic arc from C_0 to C_2. The control polygon in Figure 2.5 is approximately the shape of the Bezier curve and the curve lies within the convex hull of the Bezier polygon.

2.3.3 Cubic Bezier Curve

Bezier curve of degree three is written as:

$$B_0^3(t) = (1-t)^3$$

$$B_1^3(t) = 3t(1-t)^2$$

$$B_2^3(t) = 3t^2(1-t)$$

$$B_3^3(t) = t^3$$

$$P(t) = (1-t)^3 C_0 + 3t(1-t)^2 C_1 + 3t^2(1-t)C_2 + t^3 C_3$$

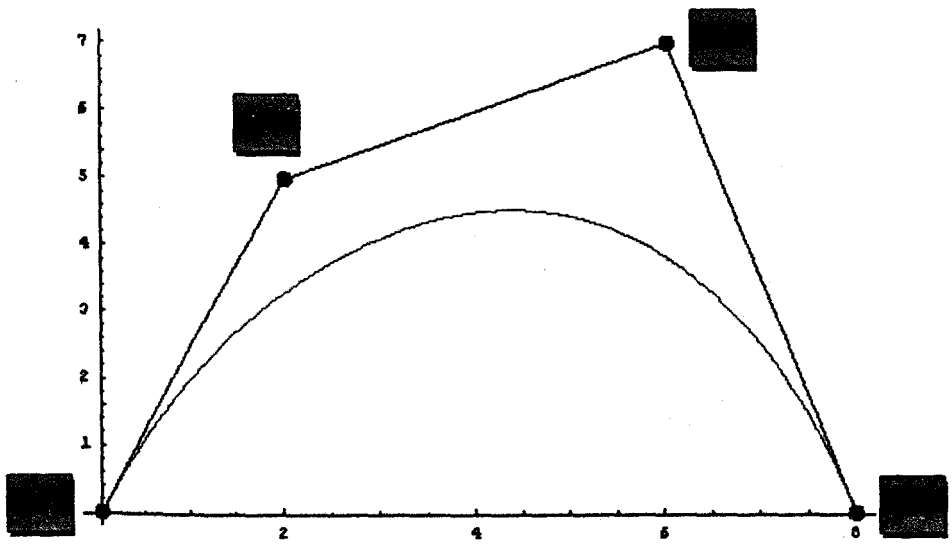

Figure 2.6: Bezier curve of degree 3

Figure 2.6 shows a parabolic arc from C_0 to C_3. The control polygon in Figure 2.6 imitates the shape of the curve and the curve is contained in the convex hull of the control polygon.

2.3.3.1 Other Examples of Cubic Bezier Curve

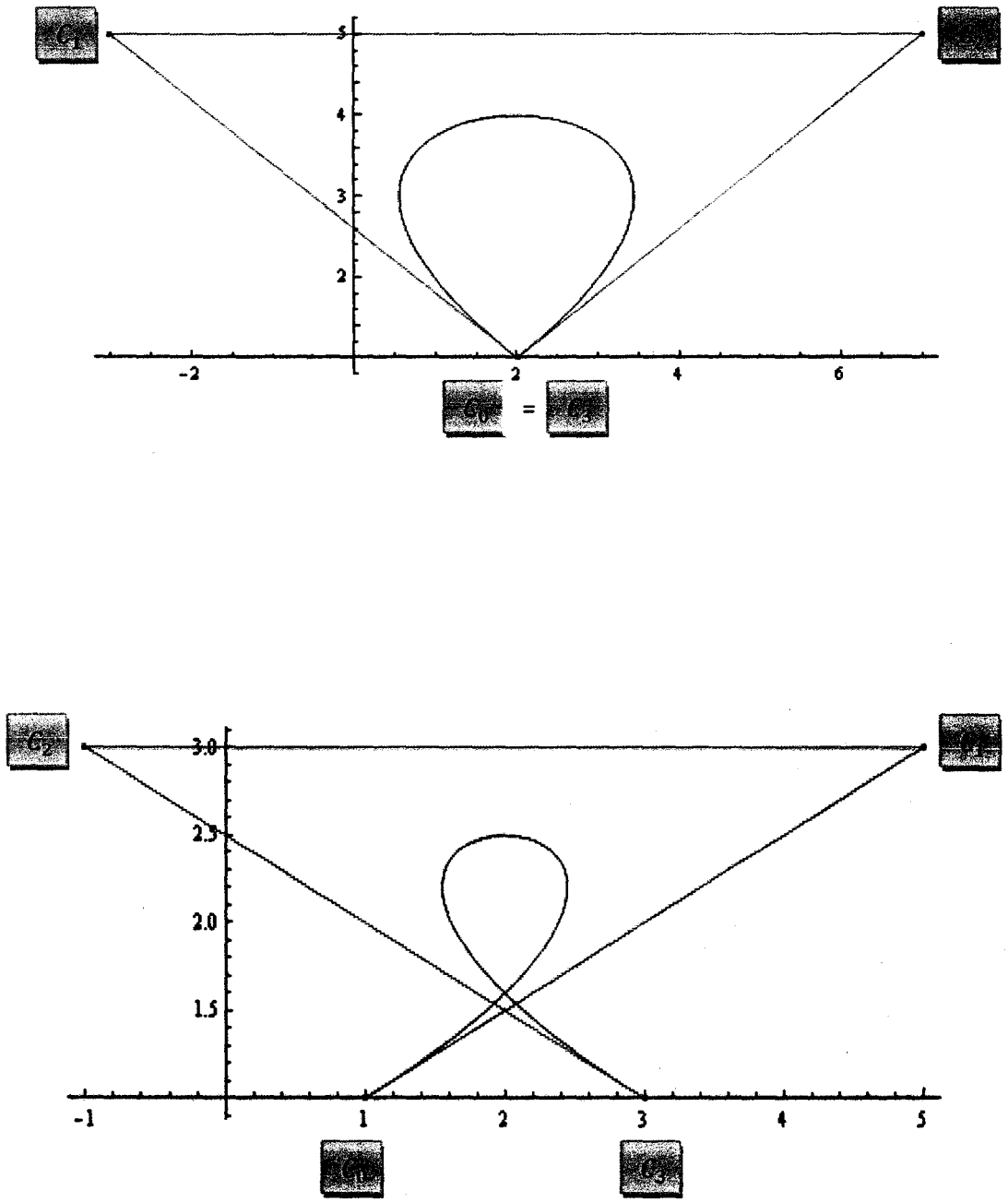

Figure 2.7: Other examples of cubic Bezier curve

2.4 Properties of Bezier Curve

The Bezier curve has the following properties:

2.4.1 Endpoint Interpolation

The Bezier curve $P(t)$ always passes through the first control point C_0 and the last control points C_n.

$$P(0) = \sum_{i=0}^{n} C_i B_i^n\ (0) = C_0,$$

$$P(1) = \sum_{i=0}^{n} C_i B_i^n\ (1) = C_n.$$

2.4.2 Designing with Bezier Polygon

The control polygon reflects the shape of the Bezier curve as in the following:

i) Convex polygon

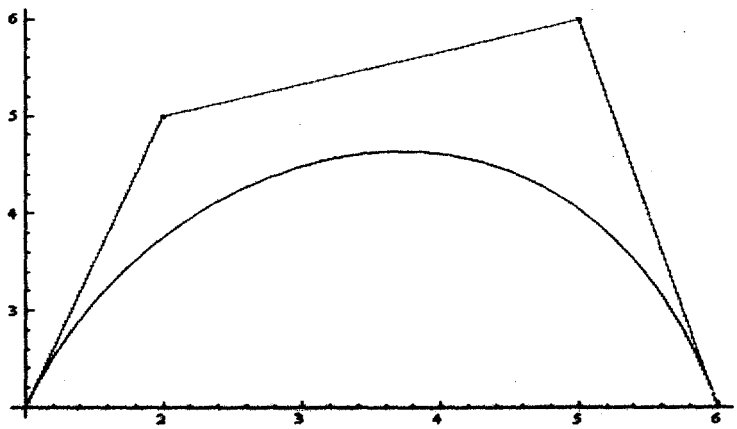

Figure 2.8: Example of Convex Bezier curve/C-shape

ii) Polygon with a point inflection

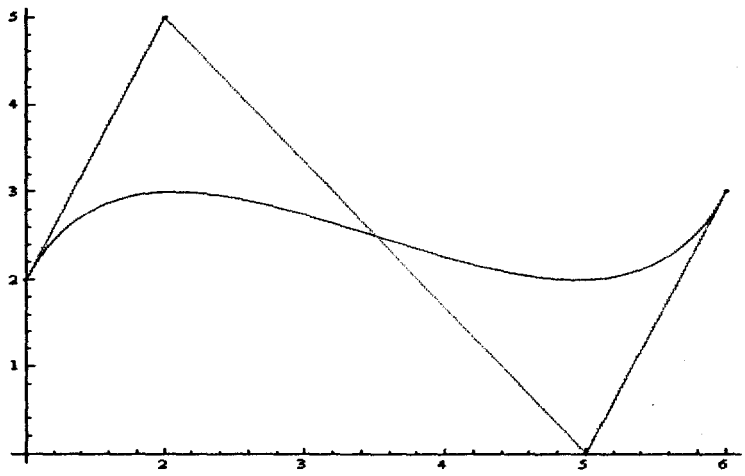

Figure 2.9: Example of Bezier curve with an inflection point /S-shape

2.4.3 Bezier Curve Derivative

The derivative of a Bernstein polynomial $B_i^n(t)$ is

$$\frac{d}{dt}B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

and the derivative of a Bezier curve $P(t)$ is:

$$\frac{d}{dt}P(t) = n\sum_{i=0}^{n} C_i(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

$$= n\left[\sum_{i=1}^{n} C_i B_{i-1}^{n-1}(t) - \sum_{i=0}^{n-1} C_i B_i^{n-1}(t)\right]$$

$$= n\left[\sum_{j=0}^{n-1} C_{j+1} B_j^{n-1}(t) - \sum_{i=0}^{n-1} C_i B_i^{n-1}(t)\right] \qquad \text{let } j = i-1,$$

$$= n\sum_{i=0}^{n-1}(C_{i+1} - C_i)B_i^{n-1}(t)$$

$$= n\sum_{i=0}^{n-1}\Delta C_i B_i^{n-1}(t)$$

where $\Delta C_i = C_{i+1} - C_i$, ($\Delta$ denotes the forward difference operator).

2.5 Matrix Formulation of Bezier Curve

A Bezier curve can be written in a matrix form by expanding the analytic definition of the curve into its Bernstein polynomial coefficients, and then writing these coefficients in a matrix form by using the polynomial power basis.

2.5.1 Quadratic Bezier Curve

The quadratic Bezier curve is $P(t) = \sum_{i=0}^{2} C_i B_i^2(t)$ or it can be re-written as:

$$P(t) = (1-t)^2 C_0 + 2t(1-t)C_1 + t^2 C_2$$

$$= \begin{bmatrix} (1-t)^2 & 2t(1-t) & t^2 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 \end{bmatrix} M \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

where $M = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

Let $T = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}$ and $C = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$. Hence, the matrix representation of quadratic Bezier curve is

$$P(t) = TMC$$

$$= \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}.$$

2.5.2 Cubic Bezier Curve

The cubic Bezier curve is $P(t) = \sum_{i=0}^{3} C_i B_i^3(t)$ or it can be re-written as:

$$P(t) = (1-t)^3 C_0 + 3t(1-t)^2 C_1 + 3t^2(1-t)C_2 + t^3 C_3$$

$$= \begin{bmatrix}(1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3\end{bmatrix} \begin{bmatrix}C_0 \\ C_1 \\ C_2 \\ C_3\end{bmatrix}$$

$$= \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} M \begin{bmatrix}C_0 \\ C_1 \\ C_2 \\ C_3\end{bmatrix}$$

where $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$.

Let $T = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix}$ and $C = \begin{bmatrix}C_0 \\ C_1 \\ C_2 \\ C_3\end{bmatrix}$. Hence, the matrix representation of cubic Bezier curve is

$$P(t) = TMC$$

$$= \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix}C_0 \\ C_1 \\ C_2 \\ C_3\end{bmatrix}.$$

It can be noted that matrix M is $(n+1) \times (n+1)$ and specifies the coefficients for the blending functions. A Bezier curve can be fully specified by the control point vector and matrix M since the form of t is known once the order of the curve has been identified.

CHAPTER 3

CONDITION FOR QUADRATIC BEZIER CURVE TO TOUCH A CONSTRAINT LINE

3.1 Introduction

This chapter explains the condition of quadratic Bezier curve so that it touches the constraint line. In general, a quadratic Bezier curve has three control points. So, only one control point of the quadratic Bezier curve has to be identified in the middle of it since the first and the last control point will then be known. That control point will be able to make the quadratic Bezier curve touch a constraint line. In order to provide a clearer explanation, the case is divided into two parts:

- Case I: The constraint line is the x-axis.
- Case II: The constraint line is any straight line.

3.2 Condition for Quadratic Bezier Curve to Touch A Constraint Line

3.2.1 Case I: The Constraint Line is the x-Axis

In this case, a discussion will be made on sufficient conditions for quadratic Bezier curve to touch the x-axis.

The first step is to define quadratic Bezier function curve as:

$$P(t) = (1-t)^2 y_0 + 2t(1-t)h + t^2 y_2 \qquad 0 \leq t \leq 1 \tag{3.1}$$

where $y_0, h, y_2 \in R^1$ are the Bezier ordinates of the curve $P(t)$. In order to find the sufficient conditions so that the curve touch the x-axis, the minimum of value h has to be identified when $y_0, y_2 \geq 0$ are provided. This is done by doing a transformation of

$$t = \frac{(x-x_0)}{(x_2-x_0)}, \tag{3.2}$$

$P(t)$ will transform into $P(x)$ where

$$P(x) = \frac{2h(x-x_0)\left(1-\frac{(x-x_0)}{(-x_0+x_2)}\right)}{-x_0+x_2} + \left(1-\frac{(x-x_0)}{(-x_0+x_2)}\right)^2 y_0 + \frac{(x-x_0)^2 y_2}{(-x_0+x_2)^2} \tag{3.3}$$

Next, differentiate $P(x)$ with respect to x. The result will be

$$P'(x) = -\frac{2(2hx-hx_0-hx_2-xy_0+x_2y_0+xy_2+x_0y_2)}{(x_0-x_2)^2}. \tag{3.4}$$

Then, solve $P'(x) = 0$, the value of x obtained will be in terms of h.

$$x = \frac{hx_0+hx_2-x_2y_0-x_0y_2}{2h-y_0-y_2} \tag{3.5}$$

Next, substitute x into (3.3),

$$P(x) = \frac{-h^2+y_0y_2}{-2h+y_0+y_2} \tag{3.6}$$

Finally, solve (3.6) making it equal to zero in order to find the value of h.

$$h = \pm\sqrt{y_0y_2} \tag{3.7}$$

Two value of h will be obtained which are $h = \sqrt{y_0y_2}$ and $h = -\sqrt{y_0y_2}$.

If $h = \sqrt{y_0y_2}$, then the curve will not touch the x-axis because h is not a minimum value. If $h = -\sqrt{y_0y_2}$, then the curve will touch the x-axis because h is a minimum value. Therefore, the condition for quadratic Bezier curve to touch a constraint line where the constraint line is the x-axis when $y_0, y_2 \geq 0$ is given as

$h = -\sqrt{y_0y_2}$.

3.2.2 Case II: The Constraint Line is Any Straight Line

As in Case I, quadratic Bezier function curve is defined as

$$P(t) = (1-t)^2 y_0 + 2t(1-t)h + t^2 y_2 \qquad 0 \leq t \leq 1 \tag{3.8}$$

where $y_0, h, y_2 \in R^1$ are the Bezier ordinates of the curve $P(t)$. Then, define the straight line as

$$G(x) = mx + c \tag{3.9}$$

where m is a tangent slope, c is a constant and $m \geq 0$. A discussion will be made on sufficient conditions for the quadratic Bezier curve to touch the straight line when $y_0, y_2 \geq 0$ are provided.

The first step is by doing a transformation of

$$t = \frac{(x-x_0)}{(x_2-x_0)}. \tag{3.10}$$

Then, the result will be $P(x)$ where

$$P(x) = \frac{2h(x-x_0)\left(1-\frac{(x-x_0)}{(-x_0+x_2)}\right)}{-x_0+x_2} + \left(1-\frac{(x-x_0)}{(-x_0+x_2)}\right)^2 y_0 + \frac{(x-x_0)^2 y_2}{(-x_0+x_2)^2} \tag{3.11}$$

Next, differentiate $P(x)$ and $G(x)$ with respect to x, $P'(x)$and $G'(x)$ will be obtained respectively.

$$P'(x) = -\frac{2(2hx-hx_0-hx_2-xy_0+x_2y_0+xy_2+x_0y_2)}{(x_0-x_2)^2} \tag{3.12}$$

and

$$G'(x) = m \tag{3.13}$$

After that, solve $P'(x) = G'(x)$ to find the value of x,

$$x = \frac{2hx_0-mx_0{}^2+2hx_2+2mx_0x_2-mx_2{}^2-2x_2y_0-2x_0y_2}{2(2h-y_0-y_2)}. \tag{3.14}$$

Then, substitute x into (3.11), and the result obtained will be

$$P(x) = \frac{-4h^2+m^2x_0{}^2-2m^2x_0x_2+m^2x_2{}^2+4y_0y_2}{4(-2h+y_0+y_2)}. \tag{3.15}$$

Next, solve $G(x) = P(x)$ in order to find the value of h where

$$h = \frac{1}{2}(2c + mx_0 + mx_2 \pm$$

$$2\sqrt{c^2 + cmx_0 + cmx_2 + m^2x_0x_2 - cy_0 - mx_2y_0 - cy_2 - mx_0y_2 + y_0y_2}) .$$

Two values of h will be obtained but the minimum value of h will have to be calculated so that the quadratic Bezier curve will touch the straight line. If the following formula is chosen

$$h = \frac{1}{2}(2c + mx_0 + mx_2 +$$

$$2\sqrt{c^2 + cmx_0 + cmx_2 + m^2x_0x_2 - cy_0 - mx_2y_0 - cy_2 - mx_0y_2 + y_0y_2}),$$

then, the curve will not touch the straight line. If

$$h = \frac{1}{2}(2c + mx_0 + mx_2 -$$

$$2\sqrt{c^2 + cmx_0 + cmx_2 + m^2x_0x_2 - cy_0 - mx_2y_0 - cy_2 - mx_0y_2 + y_0y_2}),$$

then, the curve will touch the straight line because h is a minimum. Therefore, the conditions for quadratic Bezier curve to touch any straight line when $y_0, y_2 \geq 0$ and $m \geq 0$ are given as

$$h = \frac{1}{2}(2c + mx_0 + mx_2 -$$

$$2\sqrt{c^2 + cmx_0 + cmx_2 + m^2x_0x_2 - cy_0 - mx_2y_0 - cy_2 - mx_0y_2 + y_0y_2}).$$