UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2005/2006

April/May 2006

MGM 563 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

[Arahan : *Jawab **semua empat** [4] soalan].*

...**/2-**

1. (a) If the probability density function of X is $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$ and

$$Y = \begin{cases} 1 - X^2 & \text{if } X \le 0 \\ 1 - X & \text{if } X > 0 \end{cases},$$

find the probability density function of Y, $f_Y(y)$ using the distribution function technique.

[30 marks]

(b) Assume that X_1 and X_2 are independent random variables with common probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty .$$

Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

(i) Find the probability density function of Y_1 using the moment generating function method.

(ii) Are Y_1 and Y_2 independent random variables?

[20 marks]

(c) Let $X_1, X_2, ..., X_{n+1}$ represent a random sample from a $N(0,1)$ distribution and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the distribution of the following statistics:

(i) $n\overline{X}_n^2 + X_{n+1}^2$

(ii) $\dfrac{X_{n+1}}{\sqrt{n}\,\overline{X}_n}$

(iii) $\dfrac{n\,\overline{X}_n^2}{X_{n+1}^2}$

[30 marks]

(d) If $Y \sim G(\alpha,\lambda)$, α is a positive integer and $\lambda > 0$, find the distribution of the random variable $W = 2\lambda Y$.

[20 marks]

.../3-

1. (a) *Jika fungsi ketumpatan kebarangkalian bagi X ialah* $f_X(x) = \frac{3}{8}(x+1)^2,\ -1 < x < 1$ *dan* $Y = \begin{cases} 1 - X^2 & jika\ X \le 0 \\ 1 - X & jika\ X > 0 \end{cases}$,

   cari fungsi ketumpatan kebarangkalian bagi Y, $f_Y(y)$ *dengan menggunakan teknik fungsi taburan.*

   [30 markah]

   (b) *Andaikan bahawa* X_1 *dan* X_2 *ialah pembolehubah rawak tak bersandar dengan fungsi ketumpatan kebarangkalian sepunya*

   $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ -\infty < x < \infty.$$

   Biarkan $Y_1 = X_1 - X_2$ *dan* $Y_2 = X_1 + X_2$.

   (i) *Cari fungsi ketumpatan kebarangkalian bagi* Y_1 *dengan menggunakan kaedah fungsi penjana momen.*

   (ii) *Adakah* Y_1 *dan* Y_2 *pembolehubah rawak tak bersandar?*

   [20 markah]

   (c) *Biarkan* $X_1, X_2, \ldots, X_{n+1}$ *mewakili suatu sampel rawak daripada taburan* $N(0,1)$ *dan* $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. *Cari taburan untuk statistik berikut:*

   (i) $n\overline{X}_n^2 + X_{n+1}^2$

   (ii) $\dfrac{X_{n+1}}{\sqrt{n}\,\overline{X}_n}$

   (iii) $\dfrac{n\overline{X}_n^2}{X_{n+1}^2}$

   [30 markah]

   (d) *Jika* $Y \sim G(\alpha,\lambda)$, α *ialah integer positif dan* $\lambda > 0$, *cari taburan untuk pembolehubah rawak* $W = 2\lambda Y$.

   [20 markah]

...\/4-

2. (a) If X_n follows a gamma distribution with parameters n and λ, find the limiting distribution of the random variable $Y_n = \frac{X_n}{n}$.

[20 marks]

(b) Let $Y_1, Y_2, ..., Y_n$ represent the order statistics of a random sample of size n from a distribution having a common distribution function

$$F(x) = \begin{cases} x^\alpha, & 0 < x < 1,\ \alpha > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Show that Y_i/Y_n, $i = 1, 2, \ldots, n-1$ and Y_n are independent.

[30 marks]

(c) Assume that $X_1, X_2, ..., X_n$ represent a random sample from a distribution with probability mass function

$$f(x;\theta) = \frac{1}{\theta},\ x = 1, 2, \ldots, \theta,$$

where θ is a positive integer. Find the maximum likelihood estimator of θ.

[20 marks]

(d) Let $X_1, X_2, ..., X_n$ represent a random sample from the $Be(\theta)$ distribution. If the prior distribution of Θ is given by $g_\Theta(\theta) = 1$, $0 < \theta < 1$, find the posterior Bayes estimator of θ with respect to the prior probability density function $g_\Theta(\theta)$.

[30 marks]

2. *(a) Jika X_n mengikuti taburan gama dengan parameter n dan λ, cari taburan penghad bagi pembolehubah rawak $Y_n = \frac{X_n}{n}$.*

[20 markah]

(b) Biarkan $Y_1, Y_2, ..., Y_n$ mewakili statistik tertib bagi suatu sampel rawak saiz n daripada taburan yang mempunyai fungsi taburan sepunya

$$F(x) = \begin{cases} x^\alpha, & 0 < x < 1,\ \alpha > 0 \\ 0, & \text{di tempat lain} \end{cases}$$

Tunjukkan bahawa Y_i/Y_n, $i = 1, 2, \ldots, n-1$ dan Y_n adalah tak bersandar.

[30 markah]

.../5-

(c) Andaikan bahawa $X_1, X_2, \ldots, X_n$ *mewakili suatu sampel rawak daripada taburan dengan fungsi jisim kebarangkalian*

$$f(x;\theta) = \frac{1}{\theta},\ x = 1, 2, \ldots, \theta\ ,$$

yang mana θ *ialah integer positif. Cari penganggar kebolehjadian maksimum bagi* θ.

[20 markah]

(d) Biarkan $X_1, X_2, \ldots, X_n$ *mewakili suatu sampel rawak daripada taburan* $Be(\theta)$. *Jika taburan prior bagi* Θ *diberi oleh* $g_\Theta(\theta) = 1,\ 0 < \theta < 1$, *cari penganggar Bayes posterior untuk* θ *terhadap fungsi ketumpatan kebarangkalian prior* $g_\Theta(\theta)$.

[30 markah]

3. (a) Assume that $X_1, X_2, \ldots, X_n$ represent a random sample from the Bernoulli distribution with parameter β, $0 \le \beta \le 1$. Find the Cramer-Rao lower bound for the variance of unbiased estimator of $\beta(1-\beta)$.

[30 marks]

(b) Assume that $X_1, X_2, \ldots, X_n$ represent a random sample from a $N(0,\theta)$ distribution, $\theta > 0$. Let $S = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

(i) Find Var(S).

(ii) Is S an efficient estimator of θ?

[40 marks]

(c) If X_1, X_2, X_3 represent a random sample of size 3 from a Bernoulli population with parameter p, show that $U = \frac{1}{8}(X_1 + 3X_2 + 4X_3)$ is not a sufficient estimator of p.

[20 marks]

(d) Define the exponential family of probability density functions.

[10 marks]

3. *(a) Andaikan bahawa $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan Bernoulli dengan parameter β, $0 \leq \beta \leq 1$. Cari batas bawah Cramer-Rao bagi varians penganggar saksama $\beta(1-\beta)$.*

[30 markah]

(b) Andaikan bahawa $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan $N(0,\theta)$, $\theta > 0$. Biarkan $S = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

(i) Cari Var(S).

(ii) Adakah S suatu penganggar cekap bagi θ?

[40 markah]

(c) Jika X_1, X_2, X_3 mewakili suatu sampel rawak saiz 3 daripada populasi Bernoulli dengan parameter p, tunjukkan bahawa $U = \frac{1}{8}(X_1 + 3X_2 + 4X_3)$ bukan suatu penganggar cukup untuk p.

[20 markah]

(d) Takrifkan famili eksponen untuk fungsi ketumpatan kebarangkalian.

[10 markah]

4. (a) Assume that Y_4 represents the 4^{th} ordered statistic, i.e., $Y_1 < Y_2 < Y_3 < Y_4$, for a random sample of size $n = 4$ from a uniform distribution, $U(0,\alpha)$. If k_1 and k_2, $0 < k_1 < k_2 \leq 1$ are chosen so that $P(k_1\alpha < Y_4 < k_2\alpha) = 0.95$, show that $k_1 = (0.05)^{1/4}$ and $k_2 = 1$. Find a 95% confidence interval for α.

[30 marks]

(b) State the definition of a uniformly most powerful (UMP) test.

[10 marks]

(c) Let $X_1, X_2, ..., X_n$ represent a random sample of size n from a $G(2,\alpha)$ distribution.

(i) Find the uniformly most powerful test for testing $H_0 : \alpha = 1$ vs. $H_1 : \alpha > 1$.

(ii) The following test is used for testing $H_0 : \alpha = 2$ vs. $H_1 : \alpha \neq 2$: Reject H_0 if $|\overline{X} - 1| \geq c$. Find c so that the size of the test is 0.1.

[Assume that n is sufficiently large so that the central limit theorem can be used to find an approximate value of c].

[40 marks]

.../7-

(d) Let $X_1, X_2, ..., X_n$ represent a random sample from a $N(\theta,1)$ distribution. Find the generalized likelihood-ratio test for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$.

[20 marks]

4. *(a) Andaikan bahawa Y_4 mewakili statistik tertib ke-4, yakni, $Y_1 < Y_2 < Y_3 < Y_4$, untuk suatu sampel rawak saiz $n = 4$ daripada taburan seragam, $U(0,\alpha)$. Jika k_1 dan k_2, $0 < k_1 < k_2 \leq 1$ dipilih supaya $P(k_1\alpha < Y_4 < k_2\alpha) = 0.95$, tunjukkan bahawa $k_1 = (0.05)^{1/4}$ dan $k_2 = 1$. Cari suatu selang keyakinan 95% bagi α.*

[30 markah]

(b) Nyatakan definisi ujian paling berkuasa secara seragam (UPBS).

[10 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz n daripada taburan $G(2,\alpha)$.

(i) Cari ujian paling berkuasa secara seragam untuk menguji $H_0: \alpha = 1$ lawan $H_1: \alpha > 1$.

(ii) Ujian berikut digunakan untuk menguji $H_0: \alpha = 2$ lawan $H_1: \alpha \neq 2$: Tolak H_0 jika $\left| \bar{X} - 1 \right| \geq c$. Cari c supaya saiz ujian ialah 0.1.

[Andaikan bahawa n adalah cukup besar supaya teorem had memusat dapat digunakan untuk mencari suatu nilai hampiran bagi c]

[40 markah]

(d) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan $N(\theta,1)$. Cari ujian nisbah kebolehjadian teritlak bagi menguji $H_0: \theta = \theta_0$ lawan $H_1: \theta \neq \theta_0$.

[20 markah]

.../8-

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	