

CHROMATICITY OF BIPARTITE GRAPHS WITH THREE AND FOUR EDGES DELETED

by

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KEKROMATIKAN GRAF DWI-PARTISI DENGAN TIGA DAN EMPAT SISI TERSINGKIRKAN

ABSTRAK

Graf adalah satu set titik dengan satu set garisan yang menghubungkan titik-titik. Semua titik mungkin dihubungkan dengan garisan. Pewarnaan titik adalah cara mewarnakan titik-titik sesuatu graf dengan sebilangan warna yang tertentu supaya titik yang berjiranan (di sambungkan dengan garisan) tidak mempunyai warna yang sama. Perhitungan semua cara perwarnaan yang mungkin sesuatu graf G dengan λ jenis warna akan menghasilkan satu polinomial dalam sebutan λ . Ungkapan polinomial ini adalah polinomial kromatik untuk graf G dan dilambangkan sebagai $P(G;\lambda)$. Dua graf dari satu family graf mungkin mempunyai polinomial kromatik yang sama. Graf-graf yang mempunyai polinomial kromatik yang sama adalah graf yang setara kromatik. Jika satu unsur mempunyai polinomial kromatik yang lain dari semua unsur dari setnya, graf ini adalah unik kromatik.

Graf bipartit adalah graf yang mempunyai ciri di mana set bucunya dibahagikan kepada dua set yang tidak bersilang dengan syarat bucu dalam subset-subset tersebut

tidak berjiranan. Graf bipartit lengkap adalah graf bipartit yang mempunyai ciri supaya tiap titik di bahagian berlainan adalah berjiranan dengan satu sama lain.

Projek ini adalah satu penyelidikan tentang keunikan kromatik graf bipartit lengkap dengan syarat tiga atau empat sisi dibuang.

ABSTRACT

Graphs are a set of vertices and edges. All vertices may or may not be joint. Vertex coloring is the coloring of a graph with a fixed number of colors such that adjacent vertices are of different colors. Considering all the possible ways of coloring a certain graph G with λ colors will result in a polynomial in λ . This polynomial is the chromatic polynomial of the graph G denoted by $P(G; \lambda)$. Two members from a set of graphs may have the same chromatic polynomial. These are called chromatically equivalent graphs. If a member of a set of graphs has a chromatic polynomial which is different from every other member in the set then this graph is chromatically unique.

Bipartite graphs are graphs in which the vertices are partitioned into two non intersecting sets with condition that no vertices within the sets are adjacent. A complete bipartite graph is a bipartite graph with all the vertices in each set being adjacent to all the vertices of the other set.

This project studies the chromaticity of complete bipartite graphs with three and also four edges deleted.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Graph theory is new branch of mathematics which was started around 1730 after a paper by Leonhard Euler on the subject the “Seven Bridges of Konigsberg “. In the paper Euler gave the necessary conditions for a solution and hence a definite negative solution for the puzzle of the “Seven Bridges of Konigsberg “. The “Seven Bridges of Konigsberg” is a problem which require a solution that passes through all the seven bridges only once with condition the start and the end must be at the same land mass(eulerian circuit). It must be noted that all the land masses have either three or five bridges connecting them as in the original map. Since then there are a host of famous mathematician contributing to this branch of mathematics. Another famous problem which also helps to generate a lot of interest in graph theory is the “Four Color Theorem“. Francis Guthrie first posted this problem in 1852 and since then it is still unproven (2006a).

The Four Color Theorem which states that for all planar graphs the area of the graph bounded by the edges of the graph can be colored by a minimum of only four colors with the condition that on either side of the edges will have different colors.

1.2 Fundamental Concepts

Consider a graph G which has a set of vertices and a set of lines connecting these vertices call edges. The set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. $|V(G)|$ is the order and $|E(G)|$ is the size of the graph G . Two vertices are adjacent if they are connected by an edge else they are not adjacent. Sometimes two vertices can be connected by more than one edge. Such graphs are called multigraph and the edges are called parallel edges. A loop is an edge connecting a single vertex. A simple graph is a graph with no parallel edges or loops. If all the vertices of a simple graph are adjacent to one another then the graph is a complete graph. The number of vertices adjacent to a particular vertex is called the degree of the vertex. Hence the degree of all the vertices of a complete graph of order n (K_n) is $n-1$.

A graph formed by deleting edges and or vertices of a graph G is called a subgraph of G . A spanning subgraph of G is a graph having all the vertices of G but one or a few edges deleted. A graph with k partitions of vertices such that the vertices within a partition are not adjacent to each other is called a k partite graph. Particularly if k is two then we have a bipartite graph. A walk is a sequence of alternating vertices and edges starting and ending with a vertex. If we do not repeat any edge then the walk is a trail. If we do not repeat any vertex then we call the walk a path. A path with the starting vertex same as the ending vertex we call it a circuit. Lastly a eulerian circuit is a circuit which passes through all the edges only once while a hamiltonian circuit is a circuit which visits all the vertices only once.

Vertex coloring of a graph is the coloring of a graph such that the adjacent vertices have different colors. The minimum number of colors to completely color a graph with the above condition is the chromatic number of the graph. If we use λ colors then it is a λ coloring of the graph. Depending on the graph there are many ways of coloring a graph with λ colors. Using combinatory it is possible to find the number of ways of coloring a graph in terms of λ . And this expression in λ is called the chromatic polynomial of the graph. We have other ways of finding the chromatic polynomial such as the Fundamental Reduction Theorem. Two graphs having different chromatic polynomial is defined as chromatically unique or χ -unique. Two graphs having the same chromatic polynomial are chromatically equivalent but chromatically equivalent graphs may or may not be a pair of isomorphic graphs. Two graphs G and H are isomorphic (written $G \cong H$) if there exist a one-to-one correspondence between their point set which preserves adjacency. We shall refer Harary (1972) to for all notations and terminologies not explained in this project.

1.3 Organization of Project

This project paper deals with chromaticity of complete bipartite graphs with three or four edges deleted. The objective is to use partitioning of the deleted edges to determine the chromatic equivalence or the chromatic uniqueness of the complete bipartite graph with three or four edges deleted.

In chapter two, we give a brief literature review on the chromatic equivalent graphs and chromatically unique graphs. In chapter three, we will discuss chromatically

unique of complete bipartite graphs with three edges deleted. In chapter four, we will study the chromatically unique of complete bipartite graphs with four edges deleted.

CHAPTER 2

LITERATURE REVIEW

2.1 The Chromatic Polynomial of Graphs

Graph Theory is basically the study of a set on points called vertices and the lines joining these points called edges. Since Francis Guthrie in 1852 first posted the Four Color Theorem (2006a), there have been numerous attempts in proving this theorem but up to now there is no analytical proof yet. To study this Four Color Conjecture which states that all planar graph can be color by a minimum of four colors, G.D. Birkhoff in 1912 propose the use of a function (Goh, 1987). This function is obtained by computing the number of ways of coloring a map with λ colors. It seems that solving this function will lead to prove or disprove the Four Color Conjecture but it turn out to be otherwise.

In graph coloring there are two branches namely the edge coloring and vertex coloring. In this project we are using only vertex coloring. Vertex coloring of a graph is the coloring of a graph such that the adjacent vertices have different colors. The minimum number of colors to completely color a graph with the above condition is the chromatic number of the graph. If we use λ colors then it is a λ coloring of the graph. Depending on the graph there are many ways of coloring a graph with λ colors. Using

combinatory it is possible to find the number of ways of coloring a graph in terms of λ . And this expression in λ is called the chromatic polynomial of the graph.

In 1932, Whitney used the same idea of a function derived from the number of ways of coloring the vertices of a graph using λ colors (Whitney, 1932). It turns out to be a polynomial function and hence called the chromatic polynomial of the graph denoted by $P(\lambda)$. Whitney has established many results on the chromatic polynomial of graphs. One of Whitney's contributions is the theorem that "the powers of the chromatic polynomial are consecutive and the coefficients alternate in sign" (2006b). His contributions include the systematic computation of chromatic polynomial.

Vertex coloring is the number of ways of coloring the vertices of a graph with finite number of colors given so that the adjacent vertices do not have the same color. λ – Coloring is the number of way of coloring the graph using λ different colors. For example if we have a point and λ different colors, then it is clear that we can color the graph in λ ways. Now if we have two vertices and no edge connecting the vertices then we have λ^2 ways. If the two vertices are connected by an edge then there will be only $\lambda(\lambda-1)$ ways. It should be noted that interchanging the colors between points are considered as different coloring.

To facilitate the following discussion let us denote $\lambda(\lambda-1)(\lambda-2)(\lambda-3) \dots (\lambda-k+1)$ as $\lambda^{(k)}$.

The following are a few common graphs and their number of was of coloring:

- $O(n)$ is a graph with n vertices without edges has λ^n ways.
- C_n a complete graph with n vertices has $\lambda^{(n)}$ ways.
- T_n a tree of order n has $\lambda(\lambda-1)^{n-1}$ ways.

- A star of order n has $\lambda(\lambda-1)^{n-1}$ ways.
- W_n a wheel of order n has $\lambda (\lambda-1) (\lambda-2)^{n-1}$ ways.

If we were to expand each of the expression of λ we have a polynomial in λ . And this is the chromatic polynomial of the specific graph. Below are a few examples:

- $P(C_n; \lambda) = \lambda^{(n)}$
- $P(T_n; \lambda) = \lambda(\lambda-1)^{n-1}$
- $P(W_n; \lambda) = \lambda(\lambda-1) (\lambda-2)^{n-1}$
- $P(C_3; \lambda) = \lambda^{(3)}$

$$= \lambda (\lambda-1) (\lambda-2)$$

$$= \lambda^3 - 3\lambda^2 + 2\lambda$$

For the easy regular graphs as above we can use the combinatorial methods to get the respective polynomial but for more complex graphs we have a few theorems to apply.

Theorem 2.1 Fundamental Reduction Theorem (Whitney, 1932):

Let $e = V_i V_j$ be an edge in graph G . We denote $G \setminus e$ the graph obtained from G by deleting e and by $G \cdot e$ the graph obtained from G by combining V_i to V_j and deleting all the edges incident to V_i and V_j (edge contraction).

Then $P(G; \lambda) = P(G \setminus e; \lambda) - P(G \cdot e; \lambda)$.

Lemma 2.1 (Read, 1968): If $G_1, G_2, G_3, G_4 \dots G_k$ be components of graph G then

$$P(G; \lambda) = P(G_1; \lambda) P(G_2; \lambda) P(G_3; \lambda) P(G_4; \lambda) \dots P(G_k; \lambda)$$

Lemma 2.2 (Zykor, 1949): Let G_1 and G_2 be graphs such that $G_1 \cup G_2$ be a complete graph then $P(G_1 \cup G_2; \lambda) = \frac{P(G_1; \lambda)P(G_2; \lambda)}{P(G_1 \cap G_2; \lambda)}$.

Another method of obtaining the chromatic polynomial of graphs is by using Whitney's Broken Circuit Theorem.

Some properties of the chromatic polynomial $P(G; \lambda)$ of a graph G .

Theorem 2.2 (Read, 1968):

- (1) $\deg(P(G; \lambda)) = n$
- (2) all the coefficients are integers
- (3) the leading term is λ^n
- (4) the constant term is zero
- (5) the coefficient alternate in sign
- (6) the coefficient of λ^{n-1} is $-n$
- (7) either $P(G; \lambda) = \lambda^n$ or the sum of the coefficient in $P(G; \lambda)$ is zero.

2.2 Chromatically Equivalent Graphs and Chromatically Unique Graphs

Consider a family of graphs \mathcal{G} . Each member in the set has a chromatic polynomial. If the elements of a subset of graphs are isomorphic then all members in this subset will have the same chromatic polynomial. There are some members in \mathcal{G} that have the same chromatic polynomial but are not isomorphic ($\not\cong$). These graphs are said to be chromatically equivalent.

Let graph G and H in \mathcal{G} . If $P(G; \lambda) = P(H; \lambda)$ then graph G and H are chromatically equivalent and we denote the relation as $G \sim H$. The members in \mathcal{G} can be classified into disjointed subsets with the same chromatic polynomial and these subsets are called equivalent classes or χ – equivalent classes.

Finally if in a particular χ – equivalent class which has only one member then that member is chromatically unique or χ -unique.

2.3 Chromaticity Of Bipartite Graphs

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2 . If G contains every line joining V_1 and V_2 then G is a complete bipartite graph. For any two positive integers p and q , let $K(p,q)$ denote the complete bipartite graph with $|V_1| = p$ and $|V_2| = q$.

Salzberg et al. proved that the graph $K(p, q)$ is χ -unique if $p \geq 2$ and $0 \leq q - p \leq \max\{5, \sqrt{2p}\}$; and conjectured that the graph $K(p, q)$ is χ -unique for all p, q with $2 \leq p \leq q$ (Read and Tutte, 1988). Tomescu showed that the graph $K(p, q)$ is a χ -unique if $p \geq 2$ and $0 \leq q - p \leq 2\sqrt{p+1}$ (Dong et al., 2001). Teo and Koh (1990) have proved that this is true.

Salzberg et al. also proved that the graph $K^{-1}(p, q)$ obtained by deleting any edge from $K(p, q)$ is χ -unique if $p \geq 3$ and $0 \leq q - p \leq 1$ (Read and Tutte, 1988). Teo and Koh (1990) showed the graph $K^{-1}(p, q)$ is χ -unique for all p, q with $3 \leq p \leq q$.

Peng (1991) stated these two theorems:

Theorem 2.3

The graph $K_i^{-3}(p, q)$ ($\not\cong K_5^{-3}(4, 4)$) is χ -unique for $1 \leq i \leq 6$, $p \geq 4$ and $q - p = 0$ or 1.

Theorem 2.4

The graph $K_i^{-4}(p, p+1)$ is χ -unique for $p \geq 5$ and $1 \leq i \leq 16$.

Motivated by Theorems 2.3 and 2.4, we use an alternative approach to prove the above two theorems. Our approach is by using the concept of n -independent partition in G .

CHAPTER 3

CHROMATIC UNIQUENESS OF CERTAIN BIPARTITE GRAPHS WITH THREE EDGES DELETED

3.1 Introduction

In this chapter, we shall study the chromatic uniqueness of certain bipartite graphs $K(p,q)$ with three edges deleted and the main results will be presented in section 3.3. In section 3.2, we give some known results and notations which will be used to prove our main result.

3.2 Preliminary Results and Notations

For a bipartite graph $G = (A,B;E)$ with bipartition A and B and edge set E , Let $G' = (A',B';E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

For a graph G and a positive integer k , a partition $\{A_1, A_2, \dots, A_k\}$ of $V(G)$ is called a k -independent partition in G if each A_i is a non-empty set of pairwise non-adjacent vertices.

Let $\alpha(G, k)$ denote the number of k -independent partitions in G . For any graph G of

order n , we have $P(G, \lambda) = \sum_{k=1}^{k=|V(G)|} \alpha(G, k) \lambda(\lambda-1) \dots (\lambda-k+1)$. (Dong et al., 2000).

For any bipartite graph $G=(A, B; E)$ with bipartition A and B and edge set E ,

let $\alpha'(G, 3) = \alpha(G, 3) - (2^{|A|-1} + 2^{|B|-1} - 2)$.

Let $G(A, B; E)$ be a graph in $K^{-s}(p, q)$ with $|A| = p$ and $|B| = q$. Therefore for any 3-independent partition $\{A_1, A_2, A_3\}$ in G , there are only two types of 3-independent partitions $\{A_1, A_2, A_3\}$.

Type 1: either $A_1 \cup A_2 = A, A_3 = B$ or $A_1 \cup A_3 = B, A_2 = A$.

Type 2: $A_1 \cap A \neq \emptyset, A_1 \cap B \neq \emptyset, A_2 = A - A_1$ and $A_3 = B - A_1$.

The number of 3-independent partitions of type 1 is $2^{p-1} + 2^{q-1} - 2$.

Let $\Psi(G)$ be the set of 3-independent partitions $\{A_1, A_2, A_3\}$ of type 2 in G .

Thus $|\Psi(G)| = \alpha'(G, 3)$ by the definition of $\alpha'(G, 3)$. Let $\Omega(G) = \{Q \mid Q \text{ is an independent set in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\}$. Since $s \leq q-1 \leq p-1, A-Q \neq \emptyset$ and $B-Q \neq \emptyset$ for any $Q \in \Omega(G)$. This implies that $Q \in \Omega(G)$ if and only if $\{Q, A-Q, B-Q\} \in \Psi(G)$. The following result is then obtained:

Lemma 3.1 (Read, 1988): If $G \sim H$ then $\alpha(G, k) = \alpha(H, k)$ for $k = 1, 2, \dots$

Lemma 3.2 (Dong et al., 2000): $\alpha'(G, 3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G')$.

For a bipartite graph $G = (A, B; E)$, the number of 4-independent partitions $\{A_1, A_2, A_3, A_4\}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i=1,2,3,4$ is

$$\begin{aligned} & (2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\ &= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2. \end{aligned}$$

Define

$$\alpha'(G, 4) = \alpha(G, 4) - \{(2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2\}$$

Observe that for $G, H \in K^{-s}(p, q)$,

$$\alpha(G, 4) = \alpha(H, 4) \text{ if and only if } \alpha'(G, 4) = \alpha'(H, 4).$$

Lemma 3.3 (Dong et al., 2000): For $G = \{A, B; E\} \in K^{-s}(p, q)$ with $|A| = p$ and $|B| = q$,

$$\begin{aligned} \alpha'(G, 4) &= \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) + |\{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 \\ &= \emptyset\}|. \end{aligned}$$

The following results will be used to prove our main theorems.

Theorem 3.1 (Peng, 1991): If G is χ -equivalent to $K^{-r}(p, q)$, then $G = G(p+k, q-k)$

for some k , $-h \leq k \leq \frac{1}{2}(q-p)$ where h is the largest nonnegative integer

satisfying $(q-p)h + h^2 \leq r$. In this case, G is obtained from the complete bipartite

graph $K(p+k, q-k)$ by deleting $t_k = (q-p)k - k^2 + r$ edges.

Lemma 3.4 (Peng, 1991): If $K_i^{-3}(p, q) \not\cong K_j^{-3}(p, q)$ for $p \geq 4$ and $1 \leq i \leq j \leq 6$, then $K_i^{-3}(p, q) \neq K_j^{-3}(p, q)$.

Theorem 3.2 (Peng, 1991): The graph $K_i^{-2}(p, p+2)$ ($\not\cong K_3^{-2}(3, 5)$) is χ -unique for $p \geq 3$ and $1 \leq i \leq 3$.

The following result is our main objective.

Theorem 3.3 (Peng, 1991): The graph $K_i^{-3}(p, q)$ ($\not\cong K_5^{-3}(4, 4)$) is χ -unique for $1 \leq i \leq 6$, $p \geq 4$ and $q - p = 0$ or 1 .

Proof: We consider two cases.

Case 1: $q - p = 0$.

By theorem 3.1, the only possible χ -equivalent candidates for $K_i^{-3}(p, q)$ are $K_j^{-2}(p-1, q+1)$ ($1 \leq j \leq 3$) and $K_l^{-3}(p-1, q+1)$ ($1 \leq l \leq 6$). But by Lemma 3.4, we need only to consider $K_j^{-2}(p-1, q+1)$, ($1 \leq j \leq 3$). The graph $K_j^{-2}(p-1, q+1)$, ($1 \leq j \leq 3$) is χ -unique if $K_j^{-2}(p-1, q+1) \not\cong K_3^{-2}(3, 5)$ (by theorem 3.2). Therefore $K_i^{-3}(p, q)$ ($\not\cong K_5^{-3}(4, 4)$) is χ -unique for $p \geq 4$ $1 \leq i \leq 6$. Note that $K_5^{-3}(4, 4)$ is χ -equivalent to the non-isomorphic $K_3^{-2}(3, 5)$.

Case 2: $q - p = 1$.

By theorem 3.1, the only possible χ -equivalent candidates for $K_i^{-3}(p, q)$ are $K^{-1}(p-1, q+1)$ and $K_j^{-3}(p, q)$ ($1 \leq j \leq 6$). From Lemma 3.4, we do not need to

consider $K_j^{-3}(p, q)$ ($1 \leq j \leq 6$). Also by the results of Teo and Koh (1990), the graph $K^{-1}(p-1, q+1)$ is χ -unique for $p \geq 4$. Thus the graph $K_i^{-3}(p, q)$ is χ -unique for $p \geq 4$ and $1 \leq i \leq 6$.

3.3 An Alternative Proof Of Theorem 3.3

Case 1: $q - p = 0$

From Theorem 3.1, For $q - p = 0, r = 3$,

$$0 + h^2 \leq 3, \Rightarrow h = 1 \text{ (largest non-negative integer)}$$

$$-h \leq k \leq 0,$$

$$-1 \leq k \leq 0. \Rightarrow k = -1, 0.$$

Therefore when $k = -1$, the candidate is $K_j^{-2}(p-1, q+1), 1 \leq j \leq 3$

and when $k = 0$, the candidate is $K_l^{-3}(p, q), 1 \leq l \leq 6$. The possible candidates for χ -equivalence to the graph $G \in K_i^{-3}(p, q)$, $p \geq 4$, $q - p = 0$ are $K_j^{-2}(p-1, q+1), 1 \leq j \leq 3$ and $K_l^{-3}(p, q), 1 \leq l \leq 6$.

Observe that the family $K_j^{-2}(p-1, q+1), 1 \leq j \leq 3$, consist of the following three bipartite graphs are:

$$K_1^{-2}(p, q) = K(p, q) - 2K_2,$$

$$K_2^{-2}(p, q) = K(p, q) - K(1, 2),$$

$$K_3^{-2}(p, q) = K(p, q) - K(2, 1).$$

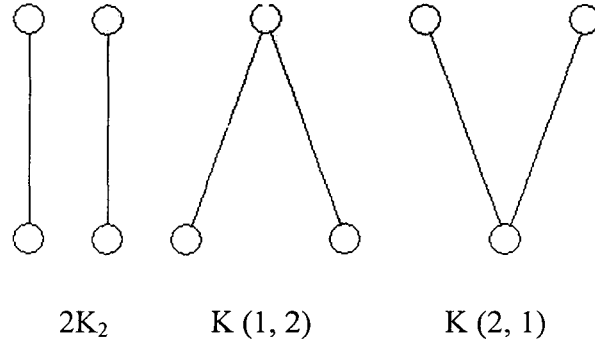


Figure 3.1 The graphs $2K_2$, $K(1,2)$ and $K(2,1)$

We can also observe that the family $K_l^{-3}(p, q), 1 \leq l \leq 6$, consist of the following six bipartite graphs:

1. $K_1^{-3}(p, q) = K(p, q) - \overline{H_1}$
2. $K_2^{-3}(p, q) = K(p, q) - \overline{H_2}$
3. $K_3^{-3}(p, q) = K(p, q) - \overline{H_3}$
4. $K_4^{-3}(p, q) = K(p, q) - \overline{H_4}$
5. $K_5^{-3}(p, q) = K(p, q) - \overline{H_5}$
6. $K_6^{-3}(p, q) = K(p, q) - \overline{H_6}$

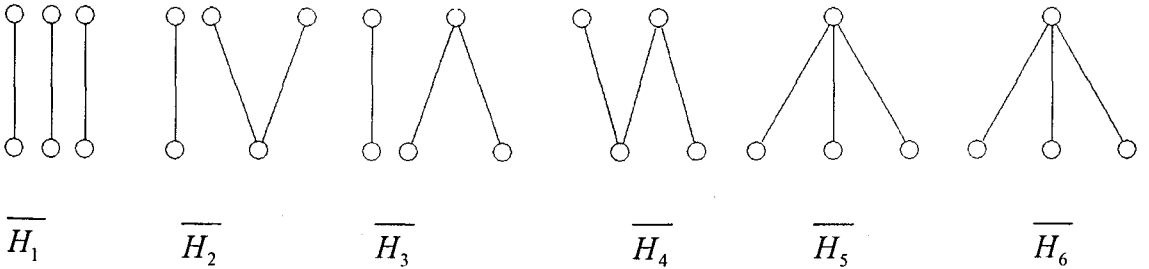
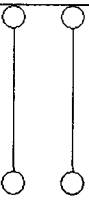
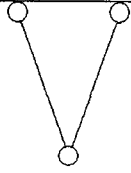


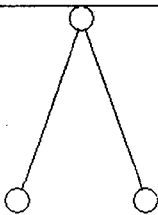
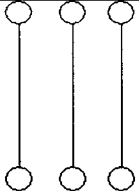
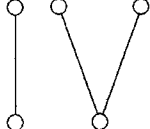
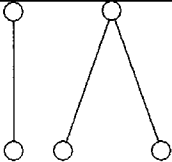
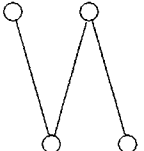
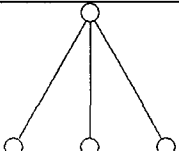
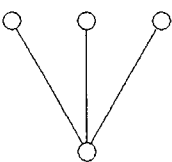
Figure 3.2 The graphs $\overline{H_i}, 1 \leq i \leq 6$

Let

1. $G_1 = K_1^{-2}(p, q) = K(p, q) - 2K_2$
2. $G_2 = K_2^{-2}(p, q) = K(p, q) - K(1, 2)$
3. $G_3 = K_3^{-2}(p, q) = K(p, q) - K(2, 1)$
4. $G_4 = K_1^{-3}(p, q) = K(p, q) - \overline{H_1}$
5. $G_5 = K_2^{-3}(p, q) = K(p, q) - \overline{H_2}$
6. $G_6 = K_3^{-3}(p, q) = K(p, q) - \overline{H_3}$
7. $G_7 = K_4^{-3}(p, q) = K(p, q) - \overline{H_4}$
8. $G_8 = K_5^{-3}(p, q) = K(p, q) - \overline{H_5}$
9. $G_9 = K_6^{-3}(p, q) = K(p, q) - \overline{H_6}$

Table 3.1 Graph G_i , $1 \leq i \leq 9$

Name of Graphs	Graphs G_i' ($G_i' = K(p, q) - G_i$)	$\alpha'(G, 3)$	$\alpha'(G, 4)$
G_1		2	$2^{p-1} + 2^{q-1} - 3$
G_2		3	$5 \cdot 2^{p-3} + 3 \cdot 2^{q-2} - 6$

G_3		3	$3 \cdot 2^{p-2} + 5 \cdot 2^{q-3} - 6$
G_4		3	$3 \cdot 2^{p-2} + 3 \cdot 2^{q-2} - 3$
G_5		4	$7 \cdot 2^{p-3} + 4 \cdot 2^{q-2} - 5$
G_6		4	$4 \cdot 2^{p-2} + 7 \cdot 2^{q-3} - 5$
G_7		5	$9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} - 9$
G_8		7	$7 \cdot 2^{p-2} + 19 \cdot 2^{q-4} - 14$
G_9		7	$19 \cdot 2^{p-4} + 7 \cdot 2^{q-2} - 14$

We group the graphs G_1 , G_2 , G_3 , G_4 , G_5 , G_6 , G_7 , G_8 , and G_9 , according to the values of $\alpha'(G_i, 3)$, which can be calculated by counting using Lemma 3.2.

Thus we have the following classification.

$$\Gamma_1 = \{G_1\} ,$$

$$\Gamma_2 = \{G_2, G_3, G_4\} ,$$

$$\Gamma_3 = \{G_5, G_6\} ,$$

$$\Gamma_4 = \{G_7\} ,$$

$$\Gamma_5 = \{G_8, G_9\} ,$$

Since Γ_1 , Γ_2 , Γ_3 , Γ_4 and Γ_5 are chromatically disjoint, the search for χ -unique graphs of family $K_i^{-3}(p, q)$, $p \geq 4$ and $q-p=0$ will be focused in the different elements in each equivalence class.

We aim to show the chromaticity of $K_i^{-3}(p, q)$, $1 \leq i \leq 6$ and thus we only consider $\Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 . To show that the graph in $K_i^{-3}(p, q)$, $1 \leq i \leq 6$ is χ -unique, it suffices to show that for every graph G_i and G_j in Γ_k , $2 \leq k \leq 5$, if $G_i \not\cong G_j$ then $\alpha'(G_i, 3) \neq \alpha'(G_j, 3)$ or $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$. The remaining work is to compare every graph in Γ_k where $2 \leq k \leq 5$.

$$(1) \quad \Gamma_2 = \{G_2, G_3, G_4\} :$$

We want to show that G_4 is χ -unique.

When $p = q$, $G_2 \cong G_3$ and we need to compare G_2 and G_4 (or G_3 and G_4).

$$\begin{aligned} \alpha'(G_2, 4) - \alpha'(G_4, 4) &= 5 \cdot 2^{p-3} + 3 \cdot 2^{q-2} - 6 - (3 \cdot 2^{p-2} + 3 \cdot 2^{q-2} - 3) \\ &= -2^{p-3} - 3 \neq 0. \end{aligned}$$

Therefore G_2, G_3 and G_4 are χ -unique.

$$(2) \quad \Gamma_3 = \{G_5, G_6\}$$

We want to show that G_5 and G_6 are χ -unique. Observe that when $p = q$, $G_5 \cong G_6$ and thus they are χ -unique.

$$(3) \quad \text{Note that } \Gamma_4 \text{ contains one graph and hence } G_7 \text{ is } \chi\text{-unique.}$$

$$(4) \quad \Gamma_5 = \{G_8, G_9\}.$$

We want to show that G_8 and G_9 are χ -unique. Observe that when $p = q$, $G_8 \cong G_9$ and hence they are χ -unique

Case 2: $q - p = 1$

By theorem 3.1, the only possible χ -equivalent candidates for $K_j^{-3}(p, q)$ are $K^{-1}(p-1, q+1)$ and $K_j^{-3}(p, q)$, $(1 \leq j \leq 6)$. By the result of Teo and Koh (1990), the graph $K^{-1}(p-1, q+1)$ is χ -unique for $p \geq 4$. Therefore we only need to consider the six bipartite graphs in $K_j^{-3}(p, q)$, $(1 \leq j \leq 6)$.

Thus we have the following classification:

$$\begin{aligned} \Gamma_1^* &= \{G_4\} \\ \Gamma_2^* &= \{G_5, G_6\} \\ \Gamma_3^* &= \{G_7\} \\ \Gamma_4^* &= \{G_8, G_9\} \end{aligned}$$

Note that Γ_1^* and Γ_3^* contain only one graph each, then G_4 and G_7 are χ -unique. The remaining work is to compare G_5 and G_6 in Γ_2^* and graphs G_8 and G_9 in Γ_4^*

When $q - p = 1$,

$$\begin{aligned}\alpha'(G_5, 4) - \alpha'(G_6, 4) &= (7 \cdot 2^{p-3} + 4 \cdot 2^{q-2} - 5) - (4 \cdot 2^{p-2} + 7 \cdot 2^{q-3} - 5) \\ &= -2^{p-3} + 2^{q-3} \\ &= 2^{p-3} \\ &\neq 0.\end{aligned}$$

Therefore G_5 and G_6 are χ -unique.

When $q - p = 1$,

$$\begin{aligned}\alpha'(G_8, 4) - \alpha'(G_9, 4) &= (7 \cdot 2^{p-2} + 19 \cdot 2^{q-4} - 14) - (19 \cdot 2^{p-4} + 7 \cdot 2^{q-2} - 14) \\ &= 9 \cdot 2^{p-4} - 9 \cdot 2^{q-4} \\ &= -9 \cdot 2^{p-4} \\ &\neq 0.\end{aligned}$$

Therefore G_8 and G_9 are χ -unique.

This completes the proof. \square

CHAPTER 4

CHROMATIC UNIQUENESS OF CERTAIN BIPARTITE GRAPHS

WITH FOUR EDGES DELETED

4.1 Introduction

In this chapter we shall study the chromatic uniqueness of certain bipartite graphs $K(p,q)$ with four edges deleted. The following theorem is from Peng (1991).

Theorem 4.1:

The graph $K_i^{-4}(p, p+1)$ is χ -unique for $p \geq 5$ and $1 \leq i \leq 16$.

In section 4.2 we shall present the possible candidates for χ -equivalence to the graph $G \in K_i^{-4}(p, q)$ and thus we discuss the alternative proof of theorem 4.1.

4.2 An Alternative Proof Of Theorem 4.1

From theorem 3.1 we found that the possible candidates for χ -equivalence to the graph $G \in K_i^{-4}(p, q)$, $p \geq 5$, $q - p = 1$ are $K_j^{-2}(p-1, q+1), 1 \leq j \leq 3$ and $K_l^{-4}(p, q), 1 \leq l \leq 16$.

For $q - p = 1$, $r = 4$,

$$h + h^2 \leq 4, \Rightarrow h = 1 \text{ (largest non-negative integer)}$$

$$-h \leq k \leq 0,$$

$$-1 \leq k \leq 0. \Rightarrow k = -1, 0.$$

Therefore when $k = -1$, the candidate is $K_j^{-2}(p-1, q+1), 1 \leq j \leq 3$


and when $k = 0$, the candidate is $K_l^{-4}(p, q), 1 \leq l \leq 16$.

From the above results, let:

1. $G_1 = K_1^{-2}(p, q) = K(p, q) - 2K_2$
2. $G_2 = K_2^{-2}(p, q) = K(p, q) - K(1, 2)$
3. $G_3 = K_3^{-2}(p, q) = K(p, q) - K(2, 1)$
4. $G_4 = K_1^{-4}(p, q) = K(p, q) - \overline{H_1}$
5. $G_5 = K_2^{-4}(p, q) = K(p, q) - \overline{H_2}$
6. $G_6 = K_3^{-4}(p, q) = K(p, q) - \overline{H_3}$
7. $G_7 = K_4^{-4}(p, q) = K(p, q) - \overline{H_4}$

8. $G_8 = K_5^{-4}(p, q) = K(p, q) - \overline{H_5}$
9. $G_9 = K_6^{-4}(p, q) = K(p, q) - \overline{H_6}$
10. $G_{10} = K_7^{-4}(p, q) = K(p, q) - \overline{H_7}$
11. $G_{11} = K_8^{-4}(p, q) = K(p, q) - \overline{H_8}$
12. $G_{12} = K_9^{-4}(p, q) = K(p, q) - \overline{H_9}$
13. $G_{13} = K_{10}^{-4}(p, q) = K(p, q) - \overline{H_{10}}$
14. $G_{14} = K_{11}^{-4}(p, q) = K(p, q) - \overline{H_{11}}$
15. $G_{15} = K_{12}^{-4}(p, q) = K(p, q) - \overline{H_{12}}$
16. $G_{16} = K_{13}^{-4}(p, q) = K(p, q) - \overline{H_{13}}$
17. $G_{17} = K_{14}^{-4}(p, q) = K(p, q) - \overline{H_{14}}$
18. $G_{18} = K_{15}^{-4}(p, q) = K(p, q) - \overline{H_{15}}$
19. $G_{19} = K_{16}^{-4}(p, q) = K(p, q) - \overline{H_{16}}$

Table 4.1 Graphs G_i , $1 \leq i \leq 19$

Graph	Graph G_i' $(G_i = K(p, q) - G_i')$	$\alpha'(G, 3)$	$\alpha'(G, 4)$
G_1		2	$2^{p-1} + 2^{q-1} - 3$