A STUDY OF SEVERAL EXPLICIT FINITE DIFFERENCE SCHEMES FOR THE LINEAR CONVECTION EQUATION

by

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Dissertation submitted in partial fulfilment
of the requirements for the degree
of Master of Science in Mathematics

May 2007

ACKNOWLEDGEMENT

First and foremost, I would like to express my deepest gratitude to Allah S.W.T who makes all things possible and gives me desire, ability, opportunity and motivation to complete this dissertation on time.

This dissertation would not have been possible without the help, advice, cooperation and commitment of various individuals.

Thanks a lot to my supervisor, Assoc. Prof. Ahmad Izani Md. Ismail. I would like to express my sincere appreciation for the opportunity given to me to learn from him. I am grateful for his exemplary professional support and guidance. His attention to detail and comments on this dissertation have been invaluable in helping me to clarify the meaning and significance of many aspects of the work and improve the overall presentation. His concern regarding in my progress has contributed, in one way or the other, to the full realization of this study within the time frame. Working with him has definitely been a rewarding experience.

I also wish to extend my thanks to my mum and all my family for their continuous support, prayers and faith in me.

Thank you.

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ABSTRAK

Perolakan adalah satu proses peralihan bendalir dari satu bahagian bendalir ke bahagian yang lain oleh pergerakan bendalir itu sendiri. Ia boleh digambarkan melalui persamaan pembezaan separa hiperbolik. Kita mengkaji persamaan perolakan linear satu dimensi yang merupakan suatu persamaan pembezaan separa linear hiperbolik satu dimensi. Di dalam disertasi ini, kita memperkenalkan beberapa kaedah pembezaan tak tersirat terhingga untuk persamaan perolakan linear. Kaedah yang digunakan ini adalah kaedah Lax-Friedrichs, kaedah Leith, kaedah Fromm, kaedah Rusanov peringkat ketiga dan kaedah Rusanov peringkat keempat. Kesemua kaedah ini akan dibincang dan dibandingkan untuk menyelesaikan persamaan perolakan yang bersifat satu dimensi. Semua kaedah yang dibentuk ini adalah berdasarkan penghampiran pembezaan terhingga. Keputusan dan hasil pengiraan ditunjukkan dan dibincangkan.

ABSTRACT

Convection is the transmission process of a constituent of a fluid from one part of a fluid to another part by a movement of the fluid itself. It can be described by hyperbolic partial differential equations. We study the one dimensional linear convection equation which is a one-dimensional linear hyperbolic partial differential equation. In this dissertation, we present several explicit finite difference schemes for the one dimensional linear convection equation. These are the Lax-Friedrichs technique, the Leith's scheme, the Fromm's technique, the Rusanov's third order scheme and the Rusanov's fourth order scheme. All these schemes will be discussed and compared for solving the one-dimensional linear convection equation problems. All these schemes also based on the weighted finite difference approximations. The results of a numerical tests are presented and discussed.

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CHAPTER 1

INTRODUCTION

1.0 Introduction

Most models of simple problems can be formulated using ordinary differential equations. However, more complicated problems of advanced physics and engineering involve working with partial differential equations (PDEs). Partial differential equations are the basis of many mathematical models of physics, chemistry and biological phenomena, and their use has also spread into economics, finance, manufacturing and other fields. It is necessary to approximate the solution of these PDEs numerically in order to investigate the predictions of the mathematical models, as exact solutions are usually unavailable or difficult to obtain.

In this dissertation, we study the linear convection equation. The linear convection equation is a one-dimensional linear partial differential equation that describes one-dimensional transmission of a constituent of a fluid from one part of the fluid to another part by the movement of the fluid itself. The one dimensional linear convection is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad (1.0.1)$$

where a, the convection velocity, is known and u is the unknown function of x and t (example; temperature or concentration). Fluid flow is a common example of convection. It moves through space.

Numerical methods are nowadays routinely usually used to solve partial differential equation. Numerical methods should not be viewed as simply tools for the purpose of application (Ang, 2006). There are many important theoretical concepts associated with numerical methods. There are two important types of numerical methods; they are the finite difference method and the finite element method. Finite difference method is straightforward as well as economical and easier to implement compared to the finite element method. In this dissertation, our focus is on the finite difference method. The main idea behind finite difference methods for obtaining the solution of a partial differential equation is to approximate the partial derivatives appearing in the equation by the definitions of a partial derivative (using function values) at a selected number of points. The (continuous) PDE is replaced by a (discrete) algebraic equation (Dehghan, 2005).

Finite difference scheme can be divided into two categories; they are implicit and explicit scheme. Implicit schemes usually have better stability properties than explicit schemes. However, explicit schemes have better own advantages, among which are their efficiency and ease of parallelization, which is becoming more and more important. In explicit schemes, we have a formula for u_i^{n+1} in terms of known values of u_i at previous time levels, whereas with an implicit scheme, we must solve the system of equations to advance to the next time level (Zhou et.al, 2000).

In this dissertation, we discuss some explicit schemes for the one dimensional linear convection equation. These schemes are the Lax-Friedrichs technique, the Leith's scheme, the Fromm's technique, the Rusanov's third order scheme and the fourth order scheme. Except for the Lax-Friedrichs technique, all of the above-mentioned schemes are not very well known in the literature.

The main objective of this dissertation is to highlight several less well known finite difference schemes for the linear convection equation and study of some of their features and characteristics corresponding to their use in solving partial differential equations.

Chapter 2 discusses about finite difference schemes. This chapter introduces finite difference approximations and it also discusses boundary conditions and theoretical concepts of consistency, stability and convergence. Chapter 3 describes the linear convection equation. In this chapter, we discuss the characteristic of a one dimensional linear convection equation and the concept of the CFL (Courant, Friedrich and Lewy) condition.

Some explicit schemes for the one dimensional linear convection equation are described in chapter 4. Chapter 4 also presents a literature review about the finite difference approximation of the convection equation. In chapter 5, we apply the schemes described in chapter 4 to solving the linear convection equation. We apply these schemes to several problems and illustrate the results in the form of tables and figures using MATLAB programming. We also discuss and compare the results obtained in this chapter. The conclusion for overall this dissertation will discuss in chapter 6.

CHAPTER 2

FINITE DIFFERENCE SCHEMES

2.0 Introduction

Generally, in this chapter, we review some aspects of the numerical approach to the solution of partial differential equations (PDE). The emphasis is on numerical methods based on the finite difference approximation scheme and their theoretical background. Much of what follows is adapted from the lecture notes of MAT 518 (2006).

2.1 Finite Difference Approximations

Suppose $u = u(x, y)$. Divide the first quadrant of the xy plane into uniform rectangles by grid lines parallel to the x axis (uniform length Δy) and grid lines parallel to the y axis (uniform length Δx). See Figure 2.1:

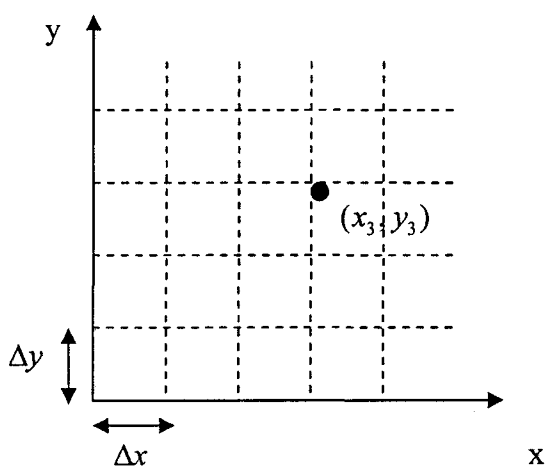

Figure 2.1: A grid point in a solution domain

The Taylor series for $u(x_i + \Delta x, y_i)$ about (x_i, y_i) is

$$u(x_i + \Delta x, y_j) = u(x_i, y_j) + \frac{\Delta x}{1!} u_x(x_i, y_j) + \frac{\Delta x^2}{2!} u_{xx}(x_i, y_j) + \frac{\Delta x^3}{3!} u_{xxx}(x_i, y_j) + \frac{\Delta x^4}{4!} u_{xxxx}(x_i, y_j) + \ldots$$

(2.1.1)

Suppose we decide to truncate the series on the right hand side (rhs) of equation (2.1.1) beginning the 3rd term. If Δx is sufficiently small, then the 4th and higher terms are much smaller than the 3rd term. We write

$$u(x_i + \Delta x, y_j) = u(x_i, y_j) + \frac{\Delta x}{1!} u_x(x_i, y_j) + O(\Delta x^2) \qquad (2.1.2)$$

$O(\Delta x^2)$ mean the sum of the truncated terms is an absolute terms at most a constant multiple of Δx^2. Divide equation (2.1.2) by Δx and rearrange it to give

$$u_x(x_i, y_j) = \frac{u(x_i + \Delta x, y_j) - u(x_i, y_j)}{\Delta x} + O(\Delta x) \qquad (2.1.3)$$

$\dfrac{u(x_i + \Delta x, y_j) - u(x_i, y_j)}{\Delta x}$ is called the *forward difference approximation* for u_x at (x_i, y_j) and is said to be first order accurate or $O(\Delta x)$ accurate. If Δx is reduced by 50%, the truncation error is reduced by 50%.

The Taylor series for $u(x_i - \Delta x, y_j)$ about (x_i, y_i) is

$$u(x_i - \Delta x, y_j) = u(x_i, y_j) - \frac{\Delta x}{1!} u_x(x_i, y_j) + \frac{\Delta x^2}{2!} u_{xx}(x_i, y_j) - \frac{\Delta x^3}{3!} u_{xxx}(x_i, y_j) + \frac{\Delta x^4}{4!} u_{xxxx}(x_i, y_j) - \ldots$$

(2.1.4)

Suppose we make a decision to truncate the series on the rhs of equation (2.1.4) beginning the 3rd term. If Δx is sufficiently small, then the 4th and higher terms are much smaller than the 3rd term. We write

$$u(x_i - \Delta x, y_j) = u(x_i, y_j) - \frac{\Delta x}{1!} u_x(x_i, y_j) + O(\Delta x^2) \qquad (2.1.5)$$

Divide equation (2.1.5) by Δx and rearrange it to yield

$$u_x(x_i, y_j) = \frac{u(x_i, y_j) - u(x_i - \Delta x, y_j)}{\Delta x} + O(\Delta x) \qquad (2.1.6.)$$

$\frac{u(x_i, y_j) - u(x_i - \Delta x, y_j)}{\Delta x}$ is called the *backward difference approximation* for u_n u_x at (x_i, y_j).

Both *forward difference approximation* and *backward difference approximation* are said to be first order accurate or $O(\Delta x)$ accurate.

If we subtract equation (2.1.4) from equation (2.1.1), we obtain

$$u(x_i + \Delta x, y_j) - u(x_i - \Delta x, y_j) = 2\Delta x u_x(x_i, y_j) + O(\Delta x^3) \qquad (2.1.7)$$

Divide equation (2.1.7) with $2\Delta x$ and rearrange it to get

$$u_x(x_i, y_j) = \frac{u(x_i + \Delta x, y_j) - u(x_i - \Delta x, y_j)}{2\Delta x} + O(\Delta x^2) \qquad (2.1.8)$$

$\frac{u(x_i + \Delta x, y_j) - u(x_i - \Delta x, y_j)}{2\Delta x}$ is called the *central difference approximation* for u_x at (x_i, y_j) and is said to be second order accurate or $O(\Delta x^2)$ accurate.

Adding (2.1.1) and (2.1.4), we obtain a similar approximation for u_{xx} at (x_i, y_j)

$$u(x_i + \Delta x, y_j) + u(x_i - \Delta x, y_j) = 2u(x_i, y_j) + \Delta x^2 u_{xx} + O(\Delta x^4) \qquad (2.1.9)$$

Divide equation (2.1.9) by Δx^2 and manipulating, gives

$$u_{xx}(x_i, y_i) = \frac{u(x_i + \Delta x, y_i) - 2u(x_i, y_i) + u(x_i - \Delta x, y_i)}{\Delta x^2} + O(\Delta x^2) \qquad (2.1.10)$$

$\dfrac{u(x_i + \Delta x, y_i) - 2u(x_i, y_i) + u(x_i - \Delta x, y_i)}{\Delta x^2}$ is called the *central difference approximation*

for u_{xx} at (x_i, y_i) and is said to be second order accurate or $O(\Delta x^2)$ accurate.

Similarly, the approximation formulas for u_y and u_{yy} are

(forward difference approximation)

$$u_y(x_i, y_j) = \frac{u(x_i, y_j + \Delta y) - u(x_i, y_j)}{\Delta y} \qquad (2.1.11)$$

(backward difference approximation)

$$u_y(x_i, y_j) = \frac{u(x_i, y_j) - u(x_i, y_j - \Delta y)}{\Delta y} \qquad (2.1.12)$$

(central difference approximation)

$$u_y(x_i, y_j) = \frac{u(x_i, y_j + \Delta y) - u(x_i, y_j - \Delta y)}{2\Delta y} \qquad (2.1.13)$$

(central difference approximation)

$$u_{yy}(x_i, y_i) = \frac{u(x_i, y_i + \Delta y) - 2u(x_i, y_i) + u(x_i, y_i - \Delta y)}{\Delta y^2}. \qquad (2.1.14)$$

2.2 Initial and Boundary Conditions

The solution of a physical problem involving a partial differential equation consists of finding a function that satisfies the PDE and the appropriate initial condition and boundary conditions.

The initial condition prescribes the unknown function throughout the given region at some initial time t, usually $t = 0$.

Boundary conditions describe the function for all time at the prescribed boundary. Two common boundary conditions are Dirichlet boundary condition and Neumann boundary condition. Dirichlet boundary condition involve the value of the function specified on the boundary while Neumann boundary condition describe the value of the derivative normal to the boundary specified on the boundary (Ang, 2006).

One of the types of boundary conditions must be specified at each point on the boundary of the closed solution domain. Different types of the boundary conditions can be specified on different portions of the boundary (Hoffman, 2001).

2.3 Theoretical Background

There are three basic theoretical concepts associated with the finite difference of PDE. These are consistency, stability, and convergence. These concepts are relevant to parabolic and hyperbolic equations, which are equations describing time evolutionary problem.

2.3.1 Consistency

The system of finite difference equation generated by discretisation process is said to be consistent with the original PDE if, in the limit that the grid spacing tends to zero, the system of finite difference equations is equivalent to the original PDE at each point. Clearly, consistency is necessary if the approximate solution is to converge to the solution of PDE under consideration (Ang, 2006). Consistency is straightforward to demonstrate although it is quite tedious and messy.

2.3.2 Stability

A numerical scheme is said to be stable if errors from any source (e.g. truncation, round-off, error in measurements) are not permitted to grow as the calculation proceeds. A scheme also is stable if its solution remains a uniformly bounded function of the initial state for all sufficiently small Δt. For time-dependent problems, stability guarantees that the method produces a bounded solution if the exact solution itself bounded (Neta, 2003).

The problem of stability is very important in numerical analysis. There are two main methods for checking the stability of linear difference equations. The first one is referred as Fourier or von Neumann method assumes the boundary conditions are periodic. The second one is called the matrix method and takes care of contributions to the error from the boundary. The von Neumann method is the more popular method in numerical scheme because it is a widely used for linear (or linearized) equations. The von Neumann method has been developed during the Second World War and was published in 1950 by Charney et al (Neta, 2003).

2.3.2 Convergence

The finite difference scheme used to approximate a differential equation is said to be convergent if the computed solution of the discretized equations tend to the exact solution of the differential equation as the grid and time spacing tend to zero.

It is difficult to show convergence directly from the definition. This is the case even for simple linear partial differential equation. In general, numerical analysis will conclude that a scheme is convergent if it is consistent and stable.

CHAPTER 3

LINEAR CONVECTION EQUATION

3.0 Introduction

In this chapter, we discuss the linear convection equation and highlight certain aspects.

3.1 One Dimensional Linear Convection Equation

Let us consider the one dimensional linear convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{3.1.1}$$

where a, the velocity is assumed to be known and u is the unknown function of x and t (e.g. temperature, concentration). The equation is a hyperbolic in partial differential equation. If a is constant and positive then the general solution can be written as

$$u(x,t) = F(x) \tag{3.1.2}$$

where the initial condition is given by

$$u(x,0) = F(x) \tag{3.1.3}$$

and $F(x)$ is known.

If $F(x)$ is specified over the complete x range, $-\infty \le x \le \infty$, the solution at some specific location (x_1, t_1) in the (x,t) plane is equal to the solution at $x_1 - at_1$ at time $t = 0$

$$u(x_1, t_1) = F(x_1 - at_1) = u(x_1 - at_1, 0) \qquad (3.1.4)$$

The solution u is constant along line AB which is a characteristic for equation (3.1.4) as illustrated in figure 3.1.

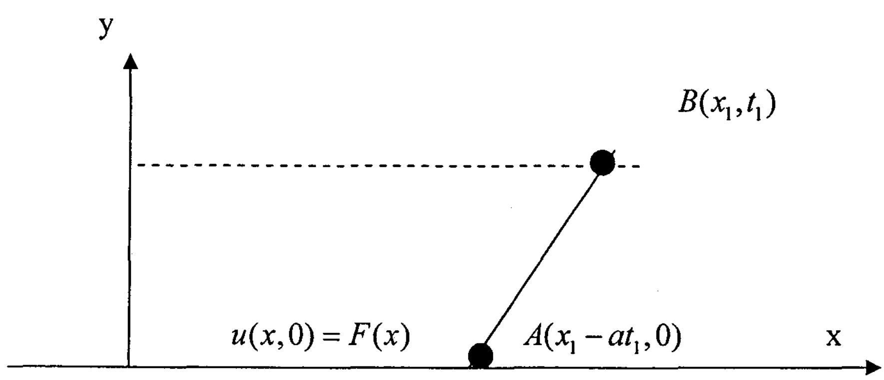

Figure 3.1: Dependence of the solution on the initial data

(Source: Fletcher (1990))

3.2 The CFL Condition

Richard Courant, Kurt Friedrichs and Hans Lewy, of the University of Gottingen in Germany, published a paper entitled "On the partial difference equations of mathematical physics". This paper, published in 1928, was written long before the invention of computers and its purpose was to apply finite difference approximations to prove existence of solutions to partial differential equations. But the "CFL" paper laid the theoretical foundations for practical finite difference computations (Trefethen, 1996). This paper became famous and has been highly cited because it identified a fundamental necessary condition for convergence of any numerical scheme which is now known as the CFL (Courant, Friedrichs and Lewy) stability condition or criterion.

The Courant number is $c = a\frac{\Delta t}{\Delta x}$, where Δx and Δt are the time and space increments, and the Courant condition is $a\frac{\Delta t}{\Delta x} \leq 1$. This condition means that a fluid particle should not travel more than one spatial step size Δx in one time step Δt..

Courant, Friedrichs and Lewy proved that the solution of the finite different system converges to that of the partial difference equation as Δx and Δt tend to the zero provided the domain of dependence of the partial different equation lies inside that of the PDE (LeVeque, 1992).

The CFL stability condition obtained can be relooked using the concept of domain of dependence (http://twister.ou.edu/CFD 2005/Chapter 3_1_2.pdf).

The solution at (x_1, t_1) depends on data in the interval $(x_1 - at_1, x_1 + t_1)$, and the domain of dependence is the area enclosed by the two characteristics lines (note here a is the convection speed).

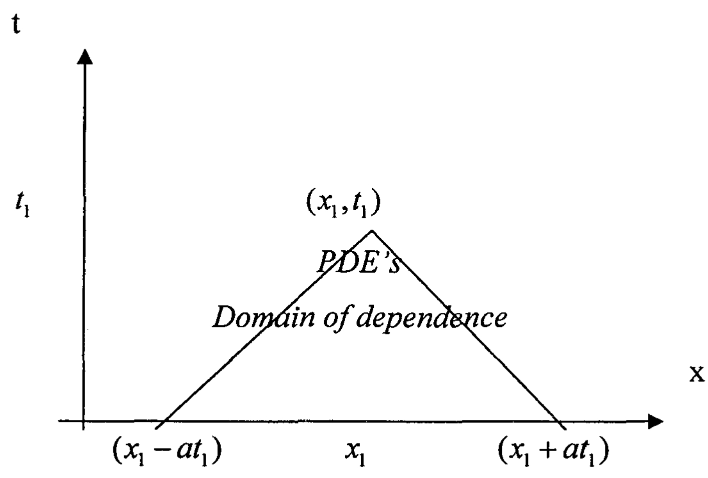

Figure 3.2.1: Domain of dependence for stability

According to http://twister.ou.edu/CFD 2005/Chapter 3_1_2.pdf , a numerical domain of dependence can be constructed as below:

Case 1 : When the numerical domain of dependence is smaller than the PDE's domain of dependence (which usually happens when Δt is large), the numerical solution cannot be expected to converge to the true solution, because the numerical solution is not using part of the initial condition. This true solution, however, is dependent on the initial values in these intervals. Different initial values there will result in different true solutions, while the numerical solution remains unaffected by their values. We therefore cannot expect the solution to match.

The numerical solution must then be unstable. Otherwise, the Lax's Equivalence theorem is (consistency + stability = convergence) is violated. The above situation occurs when $\frac{\Delta t}{\Delta x} > \frac{1}{a}$, and this results is unstable solutions.

Case II : When $\frac{\Delta t}{\Delta x} = \frac{1}{a}$, the PDE domain of dependence coincides with the numerical domain of dependence, the scheme is stable.

Case III : When $\frac{\Delta t}{\Delta x} < \frac{1}{a}$, the PDE domain of dependence is contained within the numerical domain of dependence. The numerical solution now fully depends on the initial condition. It is possible for the scheme to be stable.

Satisfaction of the CFL condition is a necessary, not a sufficient condition for stability.

The observation made by Courant, Friedrichs and Lewy was as follows: a numerical approximation cannot converge for arbitrary initial data unless it takes all of the necessary data into account. Their conclusion is that the CFL condition is necessary condition for the convergence of a numerical approximation of a PDE, linear or nonlinear (Trefethen, 1996).

CHAPTER 4

SOME EXPLICIT SCHEMES FOR THE LINEAR CONVECTION EQUATION

4.0 Introduction

In this chapter, we will consider explicit finite difference for the one dimensional linear convection equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad 0 < x < 1, \qquad 0 < t \leq T \qquad (4.0.1)$$

with initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1 \qquad (4.0.2)$$

and boundary conditions

$$u(0,t) = g_o(t),\ 0 < t \leq T \qquad (4.0.3)$$

$$u(1,t) = g_1(t),\ 0 < t \leq T \qquad (4.0.4)$$

where f, g_o and g_1 are arbitrary functions while u is unknown function. $a > 0$ is considered as a positive constant quantifying the convection process. This is the initial boundary value problem considered by Dehghan in his paper published in 2005.

4.1 Literature Review

The finite difference schemes for the linear convection equation of a partial differential equation are to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points (Dehghan, 2005). These schemes are very important in partial differential equation. The most usual way to generate these approximations is through the use of Taylor series expansion.

The domain of the problem is covered by a mesh a grid lines,

$$x_i = i\Delta x, \qquad i = 0,1,2,...,M, \qquad (4.1.1)$$

$$t_n = n\Delta t, \qquad n = 0,1,2,...,N. \qquad (4.1.2)$$

parallel to the space and time coordinate axes, respectively. The index i denotes the spatial location of a grid point, while the index n indicates the temporal step. Approximations u_i^n to $u(i\Delta x, n\Delta t)$ are calculated at the point of intersection of these lines. The constant spatial and temporal grid spacing are $\Delta x = \frac{1}{M}$ and $\Delta t = \frac{T}{N}$, respectively.

Some well known explicit schemes for the linear convection equation are the upwind scheme, Leapfrog and Lax-Wendroff, however well known implicit schemes include the Crank Nicholson and three-level fully implicit scheme. Details of their derivation, stability properties and truncation errors are given in Fletcher (1990). Generally, explicit schemes are constrained by the CFL condition and implicit schemes are unconditionally stable.

For the one-dimensional linear convection equation, solutions ranging from the exact to highly unsatisfactory can be obtained using different discretization schemes. In addition, for the same discretization scheme, the nature of the solution may be satisfactory or unsatisfactory depending on the choice of the parameter, in this case, the values of Δt and Δx. Such a sensitivity of the solution to the discretization scheme and the choice of Δt and Δx is unacceptable when dealing with a general case where the exact solution may not be known (Neta, 2003).

Zhou et.al (2000) developed explicit finite difference schemes for the convection equation. Conventional explicit finite difference schemes for the advection equations are subject to the time step restrictions dictated by the CFL condition. In many situations, time step sizes are not chosen to satisfy accuracy requirements but rather to satisfy the CFL condition. In their paper, they presented explicit algorithms which are stable far beyond the CFL restriction. The idea is to match the stencil and the real domain of dependence by characteristic analysis. However, there has not been much follow up on the work of Zhou et.al (2000).

Several high order accurate weighted based explicit finite difference schemes were discussed by Dehghan (2005) and compared for the solving the one dimensional linear convection equation. Most of the proposed numerical schemes solved the linear convection equation quite satisfactorily. The two level explicit finite difference schemes are very simple to implement and economical to use. The explicit finite difference schemes are very easy to implement for similar higher dimensional problems, but it may be more difficult when solving high dimensional problems with the implicit finite difference schemes. Dehghan (2005) remarks that, for each of the finite difference schemes investigated the modified equivalent partial differential equation is employed which permits the order of accuracy of the numerical methods to be determined. Dehghan (2005) also states that from the truncation error of the modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite difference equations that contain free weights, thus leading to more accurate methods.

In this dissertation, we apply some of the explicit schemes considered by Dehghan and apply them to a set of problems in order to compare their relative accuracy.

Dehghan (2005) considered the approximations of the derivatives in the convection equation (4.1.3) with weight θ, ϕ and γ.

$$\left.\frac{\partial u}{\partial t}\right|_i^n + a\frac{\partial u}{\partial x} = 0, \tag{4.1.3}$$

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{u_i^{n+1} - u_i^n}{\Delta t}, \tag{4.1.4}$$

$$\begin{aligned}\left.\frac{\partial u}{\partial t}\right|_i^n &\simeq \theta\frac{(u_i^n - u_{i-1}^n)}{\Delta x} + \phi\frac{(3u_i^n - 4u_{i-1}^n + u_{i-2}^n)}{2\Delta x} \\ &+ \gamma\frac{(-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n)}{12\Delta x} + (1-\phi-\theta-\gamma)\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x},\end{aligned} \tag{4.1.5}$$

The equation (4.1.5) is rearranged to obtain the weighted explicit finite difference formula:

$$\begin{aligned}u_i^{n+1} &= \frac{-c}{12}(6\phi+\gamma)u_{i-2}^n + \frac{c}{6}(3+3\theta+9\phi+\gamma)u_{i-1}^n \\ &+ \frac{-c}{6}(3-3\theta-3\phi+\gamma)u_{i+1}^n + \frac{c\gamma}{12}u_{i+2}^n + \frac{1}{2}(2-3c\phi-2c\theta)u_i^n,\end{aligned} \tag{4.1.6}$$

for $0 \le n \le N-1$, where

$$c = a\frac{\Delta t}{\Delta x} \tag{4.1.7}$$

c is the Courant number. This technique incorporates numerical diffusion with a coefficient of $\frac{a\Delta x}{2}(\theta - c)$ and $\theta \ge 0$ is a necessary (not sufficient) condition for stability (Dehghan, 2005).

4.2 The Lax-Friedrichs Technique

Consider the FTCS scheme given by equation (4.2.1):

$$u_i^{n+1} = \frac{c}{2}(u_{i-1}^n - u_{i+1}^n) + u_i^n, \qquad (4.2.1)$$

for $i = 1, 2, \ldots M - 1$.

This scheme is known to be unconditionally unstable for all values of c.

By replacing the term u_i^n in equation (4.2.1) from the FTCS scheme, with the average of the values u_{i-1}^n and u_{i+1}^n, the Lax's finite–difference formula is obtained.

$$u_i^{n+1} = \frac{1}{2}(1+c)u_{i-1}^n + \frac{1}{2}(1-c)u_{i+1}^n \qquad (4.2.2)$$

This scheme is stable in the range

$$0 < c \leq 1. \qquad (4.2.3)$$

According to Dehghan (2005), the modified equivalent partial differential equation (PDE) to the finite difference formula (4.2.2) can be written as equation (4.2.4):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \frac{a(\Delta x)(1-c^2)}{2c}\frac{\partial^2 u}{\partial x^2} + O\{(\Delta x)^2\} = 0. \qquad (4.2.4)$$

4.3 The Leith's Scheme

Setting $\theta = 0, \phi = \gamma = 0$ in equation (4.1.6) gives the following upwind type finite difference explicit technique:

$$u_i^{n+1} = \frac{1}{2}(c + c^2)u_{i-1}^n + (1 - c^2)u_i^n + \frac{1}{2}(-c + c^2)u_{i+1}^n \tag{4.3.3}$$

This is stable in the range

$$0 < c \leq 1 \tag{4.3.4}$$

The modified equivalent PDE of this method is in the following form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \sum_{q=2}^{\infty} \frac{a(\Delta x)^{q-1}}{q!} d_q \frac{\partial^q u}{\partial x^q} = 0 \tag{4.3.5}$$

The modified equivalent PDE of this method is in the following form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{a(\Delta x)^2(1 - c^2)}{6}\frac{\partial^3 u}{\partial x^3} + \frac{a(\Delta x)^3}{8}c(1 - c^2)\frac{\partial^4 u}{\partial x^4} + O\{(\Delta x)^4\} = 0 \tag{4.3.6}$$

The Leith's procedure is second order accurate (Dehghan, 2005).