UNIVERSITI SAINS MALAYSIA

Peperiksaan Semester Pertama
Sidang Akademik 2005/2006

November 2005

MAT 518 – Numerical Methods For Differential Equations
[Kaedah Berangka Untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of ***SEVEN [7]*** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **TUJUH [7]** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

*Answer **all FOUR** questions. All questions carry the same marks.*

*[Jawab **semua EMPAT** soalan. Semua soalan membawa jumlah markah yang sama.]*

1. Consider the one-dimensional transport equation $\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}-\alpha\frac{\partial^2 T}{\partial x^2}=0$ where the velocity u is known and positive.

   (a) The DuFort-Frankel scheme can be written as

$$\frac{T_j^{n+1}-T_j^{n-1}}{2\Delta t}+u\left(\frac{T_{j+1}^n-T_{j-1}^n}{2\Delta x}\right)-\alpha\frac{\left[T_{j-1}^n-\left(T_j^{n-1}+T_j^{n+1}\right)+T_{j+1}^n\right]}{\Delta x^2}=0$$

   Explain how this scheme is obtained.

   (b) Analyze the stability of the DuFort-Frankel scheme.

[100 marks]

1. Pertimbangkan persamaan pengangkutan satu dimensi $\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}-\alpha\frac{\partial^2 T}{\partial x^2}=0$ *dengan halaju* u *diketahui dan positif.*

   (a) Skema DuFort-Frankel boleh ditulis sebagai

$$\frac{T_j^{n+1}-T_j^{n-1}}{2\Delta t}+u\left(\frac{T_{j+1}^n-T_{j-1}^n}{2\Delta x}\right)-\alpha\frac{\left[T_{j-1}^n-\left(T_j^{n-1}+T_j^{n+1}\right)+T_{j+1}^n\right]}{\Delta x^2}=0$$

   Terangkan bagaimana skema ini diperoleh.

   (b) Analisis stabiliti skema DuFort-Frankel.

[100 markah]

2. Consider the one-dimensional convection equation $\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=0$ where the velocity u is known and positive. The Lax-Wendroff scheme consists of replacing $\frac{\partial T}{\partial t}$ by $\frac{T_j^{n+1}-T_j^n}{\Delta t}-0.5u^2\Delta t\frac{\partial^2 T}{\partial x^2}$

   (a) By using central differences, write down the Lax-Wendroff scheme.

   (b) Investigate the consistency of the Lax-Wendroff scheme.

[100 marks]

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2. *Pertimbangkan persamaan olahan satu dimensi* $\dfrac{\partial T}{\partial t}+u\dfrac{\partial T}{\partial x}=0$ *yang mana halaju* u *diketahui dan positif. Skema Lax-Wendroft diterbitkan dengan menggantikan* $\dfrac{\partial T}{\partial t}$ *dengan*

$$\frac{T_j^{n+1}-T_j^n}{\Delta t}-0.5u^2\Delta t\frac{\partial^2 T}{\partial x^2}.$$

(a) *Dengan menggunakan beza pusat, tulis skema Lax-Wendroff.*

(b) *Kaji kekonsistenaan skema Lax-Wendroff.*

[100 markah]

3. (a) Decide the convergence or divergence of Jacobi and Gauss-Seidel iterations for the linear solution Ax = b if

$$A=\begin{bmatrix}5 & 3 & 4\\ 3 & 6 & 4\\ 4 & 4 & 5\end{bmatrix}.$$

(b) Consider the Laplace equation

$$\nabla^2 u=0, \qquad 0<y<\frac{\pi}{2},\ 0<x<\frac{\pi}{2},$$

with values of u defined at the boundaries. Suppose that the five-point difference formula is used to discretise the equation with mesh size h, show that the spectral radius of the SOR iteration is $\rho(L_\omega)=\dfrac{1-\sin 2h}{1+\sin 2h}$.

(c) Assuming $\Delta x=\Delta y=h$, discretise the following Poisson equation using the centred difference formula:

$$-u_{xx}-u_{yy}+4u_x=x^2e^{xy}, \qquad (x,y)\in(0,1)x(0,1)$$

$$u(x,0)=u(x,1)=0, x\in(0,1)$$
$$u(0,y)=u(1,y)=0, y\in(0,1)$$

with h = $\frac{1}{4}$. Generate the linear system which arises from this discretisation using row-wise natural ordering.

[100 marks]

3. (a) *Tentukan penumpuan atau pencapahan lelaran Jacobi dan Gauss-Seidel bagi penyelesaian sistem linear $Ax = b$ jika*

$$A = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 6 & 4 \\ 4 & 4 & 5 \end{bmatrix}.$$

(b) *Pertimbangkan persamaan Laplace*

$$\nabla^2 u = 0, \qquad 0 < y < \frac{\pi}{2}, \ \ 0 < x < \frac{\pi}{2},$$

dengan nilai u tertakrif pada sempadan. Katakan rumus beza lima-titik digunakan untuk mendiskretkan persamaan ini dengan saiz mesy h, tunjukkan bahawa jejari spektrum bagi lelaran SOR ialah $\rho(L_\omega) = \dfrac{1 - \sin 2h}{1 + \sin 2h}$.

(c) *Dengan menganggapkan* $\Delta x = \Delta y = h$, *diskretkan persamaan Poisson berikut dengan menggunakan rumus beza ketengah:*

$$-u_{xx} - u_{yy} + 4u_x = x^2 e^{xy}, \qquad (x, y) \in (0,1)x(0,1)$$

$$u(x,0) = u(x,1) = 0, x \in (0,1)$$
$$u(0,y) = u(1,y) = 0, y \in (0,1)$$

dengan $h = \frac{1}{4}$. *Janakan sistem linear yang terhasil daripada pendiskretan ini dengan menggunakan tertib baris biasa.*

[100 markah]

4. (a) Consider the following elliptic problem

$$\nabla^2 u = 0, \qquad 0 < x, y < 1$$

with values of u defined at the boundaries.

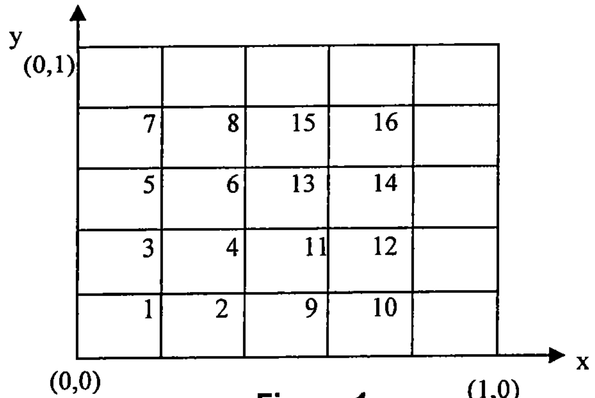

Figure 1

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i. Suppose that the five point difference formula is used to discretise the partial differential equation using the ordering as shown in Figure 1. If the resulted block matrix system is solved using the two line SOR (S.2.L.O.R) method, what is the estimated optimum relaxation parameter ω_b and the spectral radius of the S.2.L.O.R iteration matrix $\rho(L_{\omega_b})$?

ii. What is the approximate theoretical number of iterations you would expect to get if the two line S.O.R. method is used for mesh size n = 21 and tolerance $\varepsilon = 10^{-7}$. What is the rate of convergence $R_\infty(L_{\omega_b})$ of this method for this mesh size?

(b) Consider the system $A\underline{u} = \underline{b}$ where

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Generate 2 iterations of the preconditioned Conjugate Gradient method with preconditioner matrix $M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) The Gauss-Seidel iterative method in solving the system $A\underline{u} = \underline{b}$ can be written as

$$\underline{u}^{(k+1)} = (D-L)^{-1}U\underline{u}^{(k)} + (D-L)^{-1}\underline{b}$$

Using this fact, prove that the S.O.R. iterative method is defined by

$$\underline{u}^{(k+1)} = L_\omega\, \underline{u}^{(k)} + (I - \omega F)^{-1}\omega \underline{g}$$

where L_ω is the S.O.R. iterative matrix given by

$$L_\omega = (I - \omega F)^{-1}\, [\omega H + (1-\omega)I]$$

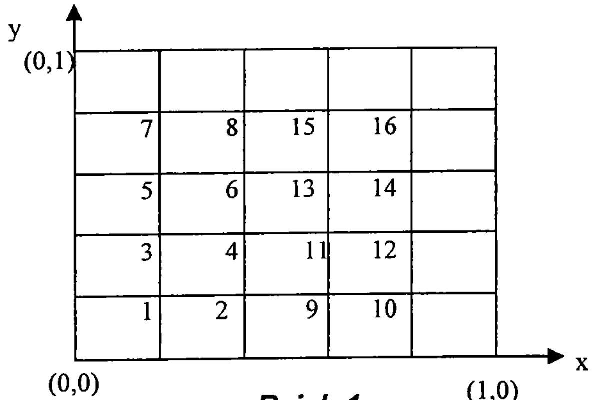
with $F = D^{-1}L,\ H = D^{-1}U,\ \underline{g} = D^{-1}\underline{b}$, where $A = D - L - U$.

[100 marks]

4. (a) *Pertimbangkan masalah eliptik berikut*

$$\nabla^2 u = 0, \qquad 0 < x, y < 1$$

dengan nilai u tertakrif pada sempadan.



Rajah 1

i. *Katakan rumus beza terhingga lima titik digunakan untuk mendiskretkan persamaan pembezaan separa ini dengan menggunakan tertib seperti yang ditunjukkan dalam Rajah 1. Sekiranya sistem matriks blok yang terhasil diselesaikan dengan menggunakan kaedah SOR (S.2.L.O.R) dua garis, apakah parameter pengenduran optimum anggaran* ω_b *dan jejari spektrum bagi matriks lelaran S.2.L.O.R dua garis,* $\rho(L_{\omega_b})$*?*

ii. *Apakah anggaran bilangan lelaran secara teori yang anda dapat jangkakan jika kaedah S.O.R. dua garis digunakan untuk saiz mesy n = 21 dan toleran* $\varepsilon = 10^{-7}$. *Apakah kadar penumpuan* $R_\infty(L_{\omega_b})$ *bagi kaedah ini untuk saiz mesy ini?*

(b) *Pertimbangkan sistem* $A\underline{u} = \underline{b}$ *di mana*

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Janakan 2 lelaran kaedah Kecerunan Konjugat berprasyarat dengan matriks prasyarat $M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) *Kaedah lelaran Gauss-Seidel dalam menyelesaikan sistem* $A\underline{u} = \underline{b}$ *boleh ditulis sebagai*

$$\underline{u}^{(k+1)} = (D-L)^{-1} U \underline{u}^{(k)} + (D-L)^{-1} \underline{b}$$

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Dengan menggunakan fakta ini , buktikan bahawa kaedah lelaran S.O.R. adalah ditakrifkan sebagai

$$\underline{u}^{(k+1)} = L_{\omega}\, \underline{u}^{(k)} + (I - \omega\, F)^{-1} \omega \underline{g}$$

di mana L_{ω} *ialah matriks lelaran S.O.R. yang diberikan oleh*

$$L_{\omega} = (I - \omega\, F)^{-1}\, [\omega H + (1 - \omega) I]$$

dengan $F = D^{-1}L,\ \ H = D^{-1}U\,,\ \ \underline{g} = D^{-1}\underline{b}\,,$ *di mana* $A = D - L - U\,.$

[100 markah]

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