UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2005/2006

November 2005

MAT 111E – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **EIGHT** pages of printed material before you begin the examination.

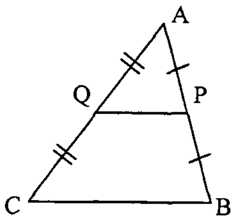
[Sila pastikan bahawa kertas peperiksaan ini mengandungi ***LAPAN*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions: Answer **all FIVE [5]** questions.

*[Arahan: Jawab **semua LIMA [5]** soalan].*

...2/-

1. (a) Refer to the following diagram. Prove, using vectors, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.



[20 marks]

(b) (i) Show that if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are three vectors then $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})$ is the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

(ii) Prove that $\boldsymbol{i}\times(\boldsymbol{u}\times\boldsymbol{i})+\boldsymbol{j}\times(\boldsymbol{v}\times\boldsymbol{j})+\boldsymbol{k}\times(\boldsymbol{u}\times\boldsymbol{k})=2\boldsymbol{u}$ where $\boldsymbol{i}=(1,0,0)$, $\boldsymbol{j}=(0,1,0)$ and $\boldsymbol{k}=(0,0,1)$.

[30 marks]

(c) (i) Find the equation of the plane Π in $\mathbb{R}^3$ that contains $P(1,3,-4)$ and parallel to the plane Π' determined by the equation $3x-6y+5z=2$.

(ii) Find an equation of the line L in $\mathbb{R}^4$ passing through $P(4,-2,3,1)$ in the direction of $\boldsymbol{u} = (2,5,-7,8)$.

[20 marks]

(d) Given $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 1 & 1\\ 0 & 7 & -20\end{bmatrix}$.

(i) Find the definition of linear transformations T and S such that $(\boldsymbol{v})T = \boldsymbol{v}.A$ and $(\boldsymbol{v})S = \boldsymbol{v}.B$, $\forall \boldsymbol{v}$.

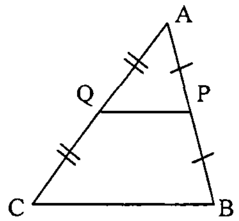
(ii) Find the definition of $3T+5S$. Use this definition to find the matrix C such that $(\boldsymbol{v})(3T+5S)=\boldsymbol{v}.C$, $\forall \boldsymbol{v}$.

(iii) Verify that $C = 3A + 5B$.

[30 marks]

...3/-

1. (a) *Rujuk gambarajah di bawah. Buktikan, menggunakan vektor, bahawa tembereng garis yang menyambungkan titik tengah dua sisi suatu segitiga adalah selari dengan sisi ketiga dan adalah separuh panjang sisi ketiga.*



[20 markah]

(b) (i) *Tunjukkan bahawa jika* $\boldsymbol{u}$, $\boldsymbol{v}$ *dan* $\boldsymbol{w}$ *adalah tiga vektor maka* $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})$ *adalah isipadu paralelepiped yang ditentukan oleh* $\boldsymbol{u}$, $\boldsymbol{v}$ *dan* $\boldsymbol{w}$.

(ii) *Buktikan bahawa* $\boldsymbol{i}\times(\boldsymbol{u}\times\boldsymbol{i})+\boldsymbol{j}\times(\boldsymbol{v}\times\boldsymbol{j})+\boldsymbol{k}\times(\boldsymbol{u}\times\boldsymbol{k})=2\boldsymbol{u}$ *dengan* $\boldsymbol{i}=(1,0,0)$, $\boldsymbol{j}=(0,1,0)$ *dan* $\boldsymbol{k}=(0,0,1)$.

[30 markah]

(c) (i) *Cari persamaan bagi satah* Π *dalam* $\mathbb{R}^3$ *yang mengandungi titik* $P(1,3,-4)$ *dan selari dengan satah* Π' *yang ditentukan oleh persamaan* $3x-6y+5z=2$.

(ii) *Cari persamaan bagi garislurus L dalam* $\mathbb{R}^4$ *yang melalui titik* $P(4,-2,3,1)$ *pada arah* $\boldsymbol{u}=(2,5,-7,8)$.

[20 markah]

(d) *Diberi* $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 3 & 5\end{bmatrix}$ *and* $B=\begin{bmatrix}-3 & 1 & 1\\ 0 & 7 & -20\end{bmatrix}$.

(i) *Cari takrif transformasi linear T dan S supaya* $(\boldsymbol{v})T=\boldsymbol{v}.A$ *dan* $(\boldsymbol{v})S=\boldsymbol{v}.B$, $\forall v$.

(ii) *Cari takrif bagi* $3T+5S$. *Gunakan takrif ini bagi mencari matriks C supaya* $(\boldsymbol{v})(3T+5S)=\boldsymbol{v}.C$, $\forall \boldsymbol{v}$.

(iii) *Tentusahkan bahawa* $C=3A+5B$.

[30 markah]

2. (a) Given that $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear transformation and that $T_{\pi,\vartheta}=\begin{bmatrix}1 & 2\\ 2 & 3\\ 3 & 4\end{bmatrix}$ where $\pi=\{e_1+e_2, e_2+e_3, e_3\}$ and $\vartheta=\{e_1,e_2\}$. Find the definition for T.

[20 marks]

...4/-

(b) Let the function $T:\mathbb{R}^4 \to \mathbb{R}^2$ be defined by $(x,y,z,t)T=(x+y+z,z+t)$.

(i) Show that T is a linear transformation.

(ii) Find *Ker T* and *Im T*, then verify the Dimension Theorem.

[40 marks]

(c) Given that $T:\mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation. If $\Omega=\{v_1,v_2,....,v_n\}$ is a basis of $\mathbb{R}^n$, show that $\Psi=\{v_1T,v_2T,....,v_nT\}$ is a basis of *Im T.*

[20 marks]

(d) Solve

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

by Gaussian elimination and back-substitution.

[20 marks]

2. (a) *Diberi* $T:\mathbb{R}^3 \to \mathbb{R}^2$ *adalah suatu transformasi linear dan* $T_{\pi,\vartheta}=\begin{bmatrix}1 & 2\\ 2 & 3\\ 3 & 4\end{bmatrix}$ *dengan* $\pi=\{e_1+e_2,e_2+e_3,e_3\}$ *and* $\vartheta=\{e_1,e_2\}$. *Dapatkan takrif bagi T.*

[20 markah]

(b) *Biar fungsi* $T:\mathbb{R}^4 \to \mathbb{R}^2$ *ditakrifkan dengan* $(x,y,z,t)T=(x+y+z,z+t)$.

(i) *Tunjukkan bahawa T adalah transformasi linear.*

(ii) *Cari Ker T dan Im T, kemudian tentusahkan Teorem Dimensi.*

[40 markah]

(c) *Diberi* $T:\mathbb{R}^n \to \mathbb{R}^m$ *adalah transformasi linear satu-ke-satu. Jika* $\Omega=\{v_1,v_2,....,v_n\}$ *adalah asas bagi* $\mathbb{R}^n$, *tunjukkan bahawa* $\Psi=\{v_1T,v_2T,....,v_nT\}$ *adalah asas bagi Im T.*

[20 markah]

(d) *Selesaikan*

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

menggunakan penghapusan Gauss dan penggantian-belakang.

[20 markah]

...5/-

3. (a) Find the rank of the matrix

$$A=\begin{bmatrix}1 & 2 & 1 & 2 & 3 & 1\\ 2 & 4 & 3 & 7 & 7 & 4\\ 1 & 2 & 2 & 5 & 5 & 6\\ 3 & 6 & 6 & 15 & 14 & 15\end{bmatrix}$$

[Hint: Find the column rank of A]

[20 marks]

(b) Given that B is an $n \times m$ matrix and that $T:\mathbb{R}^n \to \mathbb{R}^m$ is defined by $(v)T = v.B$.

(i) If $\rho(B)=n$, show that T is one-to-one.

(ii) If $\rho(B)=m$, show that T is onto.
[Note: $\rho(B)$ is the rank of B]

[30 marks]

(c) Given $W=\left\{(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} a_i = 0\right\}$. Under the same addition and scalar multiplication as defined in $\mathbb{R}^n$, show that W is a subspace of $\mathbb{R}^n$.

[20 marks]

(d) Show that if $U_1, U_2, U_3, \ldots, U_n$ are subspaces of a vector space V, then $U_1 \cap U_2 \cap U_3 \cap \ldots \cap U_n$ is also a subspace of V.

[30 marks]

3. (a) Cari pangkat bagi matriks

$$A=\begin{bmatrix}1 & 2 & 1 & 2 & 3 & 1\\ 2 & 4 & 3 & 7 & 7 & 4\\ 1 & 2 & 2 & 5 & 5 & 6\\ 3 & 6 & 6 & 15 & 14 & 15\end{bmatrix}$$

[Petunjuk: Cari pangkat lajur A]

[20 markah]

(b) Diberi B suatu matriks $n \times m$ dan $T:\mathbb{R}^n \to \mathbb{R}^m$ ditakrif sebagai $(v)T = v.B$.

(i) Jika $\rho(B)=n$, tunjukkan bahawa T adalah satu ke satu.

(ii) Jika $\rho(B)=m$, tunjukkan bahawa T adalah keseluruhan.
[Nota: $\rho(B)$ ialah pangkat B]

[30 markah]

...6/-

(c) *Diberi* $W = \left\{ (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} a_i = 0 \right\}$. *Di bawah operasi penambahan dan pendaraban skalar yang sama seperti yang tertakrif bagi* $\mathbb{R}^n$, *tunjukkan bahawa* W *adalah subruang* $\mathbb{R}^n$.

[20 markah]

(d) *Tunjukkan bahawa jika* $U_1, U_2, U_3, \ldots, U_n$ *adalah subruang suatu ruang vektor* V, *maka* $U_1 \cap U_2 \cap U_3 \cap \ldots \cap U_n$ *adalah juga subruang* V.

[30 markah]

4. (a) Given $S = \{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}$.

   (i) Find $\mathcal{L}(S)$.

   (ii) Let $U = \mathcal{L}(S)$, show that $\dim U = 3$.

   (iii) Suppose $W = \{(c, 2d, d, c) \mid c, d \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$. Determine the basis and dimension of $U \cap W$.

[30 marks]

(b) Given a vector space V containing the subsets S and T. Show that:

   (i) If $S \subseteq T$, then $\mathcal{L}(S) \subseteq \mathcal{L}(T)$.

   (ii) $\mathcal{L}(S) = S$ iff S is a subspace of V.

[20 marks]

(c) Given

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Use the Cayley-Hamilton Theorem to find A^{-1}.
[Note: Other methods used will not be given marks]

[20 marks]

(d) Find the best approximate solution to the linear system

$$\begin{aligned} x - y &= 4 \\ 3x + 2y &= 1 \\ -2x + 4y &= 3 \end{aligned}$$

[30 marks]

...7/-

4. (a) *Diberi* $S = \{(1,0,0,0),(0,1,0,-1),(0,0,1,2)\}$.

   (i) *Cari* $\mathcal{L}(S)$.

   (ii) *Biar* $U = \mathcal{L}(S)$, *tunjukkan bahawa* $\dim U = 3$.

   (iii) *Andai* $W = \{(c,2d,d,c) \mid c,d \in \mathbb{R}\}$ *adalah suatu subruang* $\mathbb{R}^4$. *Dapatkan asas dan dimensi bagi* $U \cap W$.

[30 markah]

(b) *Diberi V suatu ruang vektor yang mengandungi S and T. Tunjukkan bahawa:*

   (i) *Jika* $S \subseteq T$, *maka* $\mathcal{L}(S) \subseteq \mathcal{L}(T)$.

   (ii) $\mathcal{L}(S) = S$ *jika hanya jika S adalah subruang V.*

[20 marks]

(c) *Diberi*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Gunakan Teorem Cayley-Hamilton untuk mendapatkan A^{-1}.
[Nota: Kaedah lain tidak akan diberikan markah]

[20 markah]

(d) *Cari penyelesaian hampiran terbaik bagi sistem*

$$\begin{aligned} x - y &= 4 \\ 3x + 2y &= 1 \\ -2x + 4y &= 3 \end{aligned}$$

[30 markah]

5. (a) Given *S*, *T*, *U* are subsets of a vector space *V*. Prove the following:

   (i) $S \subseteq (S^{\perp})^{\perp}$.

   (ii) If $T \subseteq U$, then $U^{\perp} \subseteq T^{\perp}$.

   (iii) $S^{\perp} \subseteq \mathcal{L}(S)^{\perp}$.

   [Note: $S^{\perp}$, $U^{\perp}$ and $T^{\perp}$ are orthogonal complements of *S*, *U* and *T* respectively]

[30 marks]

(b) Suppose $\{u_1, u_2, \ldots, u_r\}$ is an orthogonal set of vectors.
Prove that

$$\| u_1 + u_2 + \ldots + u_r \|^2 = \| u_1 \|^2 + \| u_2 \|^2 + \ldots + \| u_r \|^2.$$

[20 marks]

...8/-

(c) Use the Gram-Schmidt process to find an orthonormal basis for the subspace spanned by $\boldsymbol{u} = (1,2,-2)$ and $\boldsymbol{v} = (-1,-3,1)$.

[30 marks]

(d) Given that $\{v_1, v_2, ..., v_s\}$ is an orthonormal set in a subspace V of $\mathbb{R}^n$. Show that there exist vectors $w_{s+1}, w_{s+2}, ..., w_n$ in V so that $\{v_1, v_2, ..., v_s, w_{s+1}, w_{s+2}, ..., w_n\}$ is an orthonormal basis of V.

[20 marks]

5. *(a) Diberi S, T, U adalah subset-subset dari ruang vektor V. Buktikan yang berikut:*

(i) $S \subseteq (S^{\perp})^{\perp}$.

(ii) Jika $T \subseteq U$, *maka* $U^{\perp} \subseteq T^{\perp}$.

(iii) $S^{\perp} \subseteq \mathcal{L}(S)^{\perp}$.

[Nota: $S^{\perp}$, $U^{\perp}$ and $T^{\perp}$ adalah pelengkap ortogonal bagi S, U and T]

[30 markah]

(b) Andai $\{u_1, u_2, ..., u_r\}$ *adalah set vektor-vektor ortogonal. Buktikan bahawa*

$$\| u_1 + u_2 + \ldots + u_r \|^2 = \| u_1 \|^2 + \| u_2 \|^2 + \ldots + \| u_r \|^2 .$$

[20 markah]

(c) Gunakan proses Gram-Schmidt untuk mencari suatu asas ortonormal untuk subruang yang direntang oleh $\boldsymbol{u} = (1,2,-2)$ *and* $\boldsymbol{v} = (-1,-3,1)$.

[30 markah]

(d) Diberi $\{v_1, v_2, ..., v_s\}$ *adalah suatu set ortonormal dalam subruang V dari* $\mathbb{R}^n$. *Tunjukkan bahawa wujud vektor-vektor* $w_{s+1}, w_{s+2}, ..., w_n$ *dalam V supaya* $\{v_1, v_2, ..., v_s, w_{s+1}, w_{s+2}, ..., w_n\}$ *adalah asas ortonormal V.*

[20 markah]

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