UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2005/2006

Jun 2006

MAT 102E – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of <u>SIX</u> pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>ENAM</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions :   Answer **all four** [4] questions.

*[Arahan   :   Jawab **semua empat** [4] soalan.]*

1. (a) The sequence $\{a_n\}$ is defined as

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}.$$

    (i)   Show that $\{a_n\}$ is decreasing.

    (ii)   Show that $\lim\limits_{n \to \infty} a_n$ exists and that this limit lies between $\frac{1}{2}$ and 1.

    (iii)   Is the series $\sum\limits_{n=1}^{\infty} a_n$ convergent or divergent ? Give your reason.

  (b) (i)   If $\sum\limits_{n=0}^{\infty} a_n$ converges, prove that $\lim\limits_{n \to \infty} a_n = 0$.

    (ii)   Is the converse of (i) true?
          Prove it if it is true, otherwise give an example to show that it is not true.

  (c)   Determine whether each of the following series converges or diverges.

    (i) $\sum\limits_{k=1}^{\infty} \dfrac{k+2}{k^2 - 5k + 3}$

    (ii) $\sum\limits_{k=1}^{\infty} \dfrac{(k+1)^{\frac{1}{2}}}{k^{\frac{1}{2}} + 4}$

    (iii) $\sum\limits_{k=3}^{\infty} \dfrac{1 + (\ln k)^3}{k \, (\ln k)^5}$

                                    [100 marks]

1. (a) Jujukan $\{a_n\}$ ditakrifkan sebagai

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}.$$

    (i)   Tunjukkan bahawa $\{a_n\}$ adalah menyusut.

    (ii)   Tunjukkan bahawa $\lim\limits_{n \to \infty} a_n$ wujud dan had ini terletak di antara $\frac{1}{2}$ dan 1.

    (iii)   Adakah siri $\sum\limits_{n=1}^{\infty} a_n$ menumpu atau mencapah? Berikan alasan.

  (b) (i)   Jika $\sum\limits_{n=0}^{\infty} a_n$ menumpu, buktikan bahawa $\lim\limits_{n \to \infty} a_n = 0$.

    (ii)   Adakah akas (i) benar?
          Buktikannya jika ia benar atau berikan satu contoh untuk menunjukkan bahawa ia tidak benar.

(c)   Tentukan sama ada setiap siri yang berikut menumpu atau mencapah.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{k+2}{k^2-5k+3}$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)^{\frac{1}{2}}}{k^{\frac{1}{2}}+4}$

(iii) $\displaystyle\sum_{k=3}^{\infty} \frac{1+(\ln k)^3}{k\,(\ln k)^5}$

[100 markah]

2. (a) Find the following limits :

(i) $\displaystyle\lim_{x\to 0} \frac{x-x\,e^{3x}}{1-\cos 3x}$

(ii) $\displaystyle\lim_{x\to\infty} \left(2x+e^x \right)^{\frac{1}{x}}$

(iii) $\displaystyle\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n} \right)$

(x is a real number and n is a positive integer)

(b) Use the squeezing principle to show that

$$\lim_{n\to\infty} \frac{6^n}{n!} = 0.$$

Hence, find $\displaystyle\lim_{n\to\infty} \frac{n!+6^n+1}{n!+6^{n+1}}$.

(c) Given $f(x,y) = \begin{cases} \dfrac{3x^3-y^2}{x^2+3y^2}, & (x,y)\neq(0,0) \\[2mm] 0, & (x,y)=(0,0) \end{cases}$.

(i)   Determine whether $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

(ii)   Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

(iii)   Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(x,y)\neq(0,0)$.

(iv)   At the point $(1,1)$, find the maximum value of the directional derivative and the direction in which it occurs.

[100 marks]

2. *(a) Cari had yang berikut:*

(i) $\displaystyle\lim_{x\to 0} \frac{x-x\,e^{3x}}{1-\cos 3x}$

(ii) $\lim\limits_{x \to \infty} \left(2x + e^x \right)^{\frac{1}{x}}$

(iii) $\lim\limits_{n \to \infty} \left(\sqrt{n + \sqrt{n}} - \sqrt{n} \right)$

(x merupakan nombor nyata dan n ialah integer positif)

(b)   Gunakan prinsip tersepit untuk menunjukkan bahawa

$$\lim\limits_{n \to \infty} \frac{6^n}{n!} = 0.$$

Dengan ini, cari $\lim\limits_{n \to \infty} \dfrac{n! + 6^n + 1}{n! + 6^{n+1}}$.

(c)   Diberi $f(x, y) = \begin{cases} \dfrac{3x^3 - y^2}{x^2 + 3y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$.

(i)   Tentukan sama ada $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ wujud.

(ii)   Cari $\dfrac{\partial f}{\partial x}(0, 0)$ dan $\dfrac{\partial f}{\partial y}(0, 0)$.

(iii) Cari $\dfrac{\partial f}{\partial x}$ dan $\dfrac{\partial f}{\partial y}$ pada $(x, y) \neq (0, 0)$.

(iv)   Pada titik $(1, 1)$, cari nilai maksimum terbitan berarah dan arah yang padanya maksimum ini berlaku.

[100 markah]

3. (a)   Given that $f(x, y) = 3x^2 + 3y^2 - 4x$, $(x, y) \in \Re^2$.
    (i)   Classify the critical points of f.
    (ii) Find the extremum of f on the domain D where
          $D = \{(x, y) \in \Re^2 \mid x^2 + y^2 \leq 1, \ y \geq 0\}$.

    (b)   If $V = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that
          $$\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 = \left(\frac{\partial V}{\partial r} \right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \theta} \right)^2.$$

    (c)   Determine whether each of the following improper integrals converges.
          (i) $\displaystyle\int_5^\infty \frac{\sqrt{x^8 + 100}}{(2x^2 - x + 6)^3}\, dx$

          (ii) $\displaystyle\int_0^{4^-} \frac{1}{16 - x^2}\, dx$

[100 marks]

3. (a) *Diberi bahawa* $f(x, y) = 3x^2 + 3y^2 - 4x$, $(x, y) \in \Re^2$.

    (i) *Kelaskan titik genting bagi* f.

    (ii) *Cari ekstremum* f *pada domain* D *di mana*

$$D = \{(x, y) \in \Re^2 \mid x^2 + y^2 \leq 1, \; y \geq 0\}.$$

  (b) *Jika* $V = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, *tunjukkan bahawa*

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \theta}\right)^2.$$

  (c) *Tentukan bahawa setiap kamiran tak wajar yang berikut adalah menumpu.*

    (i) $\displaystyle\int_5^\infty \frac{\sqrt{x^8 + 100}}{(2x^2 - x + 6)^3}\, dx$

    (ii) $\displaystyle\int_0^{4^-} \frac{1}{16 - x^2}\, dx$

                                         [100 markah]

4. (a) Given that $f(x) = \sin x$.

    (i) Find the Taylor's polynomial $T_2\left(f, \dfrac{\pi}{3}\right)$ of degree two about $\dfrac{\pi}{3}$ for f

        and the corresponding remainder term $R_2\left(f, \dfrac{\pi}{3}\right)$.

    (ii) Hence, approximate $\sin 31°$ by using $T_2\left(f, \dfrac{\pi}{3}\right)$

        and compute the corresponding absolute error bound.

  (b) Evaluate the integrals :

    (i) $\displaystyle\iint_T (e^y - 1)\, dx\, dy$

        where T is the trapezium with vertices $(0, 0), (1, 0), (0,1)$ and $(1, 2)$.

    (ii) $\displaystyle\int_0^3 \int_{x^2}^9 x \cos y^2 \, dy\, dx$

    (iii) $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (8 - x^2 - y^2)^{-\frac{1}{2}}\, dx\, dy$

  (c) Find the volume of the solid which is bounded by the three coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $3x + 2y + z = 6$.

                                         [100 marks]

4. (a) Diberi bahawa $f(x) = \sin x$.

    (i) Cari polinomial Taylor $T_2\left(f, \dfrac{\pi}{3}\right)$ berdarjah dua di sekitar $\dfrac{\pi}{3}$ bagi f

    dan sebutan baki $R_2\left(f, \dfrac{\pi}{3}\right)$ yang sepadan.

    (ii) Dengan ini, anggarkan $\sin 31°$ dengan menggunakan $T_2\left(f, \dfrac{\pi}{3}\right)$

    dan cari batas bagi mutlak ralat yang sepadan.

(b) Nilaikan kamiran:

    (i) $\displaystyle\iint_T (e^y - 1) \, dx \, dy$

    di mana T ialah trapezium dengan bucu $(0, 0), (1, 0), (0,1)$ and $(1, 2)$.

    (ii) $\displaystyle\int_0^3 \int_{x^2}^9 x \cos y^2 \, dy \, dx$

    (iii) $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (8 - x^2 - y^2)^{-\frac{1}{2}} \, dx \, dy$

(c) Cari isipadu bongkah yang dibatasi oleh tiga satah koordinat
$x = 0$, $y = 0$, $z = 0$ dan satah $3x + 2y + z = 6$.

                                [100 markah]

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