UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2005/2006

April/Mei 2006

MAT 102E – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini].

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan].*

1. (a) Find the following limits of functions or sequences :

(i) $\lim\limits_{x \to 0} \dfrac{\ell n\, x}{\ell n(2e^x - 2)}$

(ii) $\lim\limits_{x \to 0^+} \left(\cos x + \sin x\right)^{\frac{1}{x}}$

(iii) $\lim\limits_{n \to \infty} \left(\sqrt{n^4 + 3n^2} - n^2\right)$

(iv) $\lim\limits_{n \to \infty} \dfrac{3^n\left(\sqrt{2} + (-1)^n\right)}{n!}$

(x is a real number and n is a positive integer)

(b) The sequence $\{ a_n \}$ is defined as

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3}\left(2a_n + 5\right), \quad \forall\, n \geq 1.$$

(i) Determine whether $\{ a_n \}$ is increasing or decreasing.

(ii) Is $\{ a_n \}$ bounded above ?

(iii) Does $\lim\limits_{n \to \infty} a_n$ exist ? If yes, find this limit.

(iv) Is the series $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^2}$ convergent or divergent ?

(Give reasons to support your answers)

(c) If the limit of a sequence $\{ b_n \}$ exists, prove that the limit is unique.

[100 marks]

1. (a) Cari had fungsi atau jujukan yang berikut :

(i) $\lim\limits_{x \to 0} \dfrac{\ell n\, x}{\ell n(2e^x - 2)}$

(ii) $\lim\limits_{x \to 0^+} \left(\cos x + \sin x\right)^{\frac{1}{x}}$

(iii) $\lim\limits_{n \to \infty} \left(\sqrt{n^4 + 3n^2} - n^2\right)$

(iv) $\lim\limits_{n \to \infty} \dfrac{3^n\left(\sqrt{2} + (-1)^n\right)}{n!}$

(x ialah nombor nyata dan n ialah integer positif)

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(b) *Jujukan* $\{ a_n \}$ *ditakrifkan sebagai*

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3}\left(2a_n + 5\right), \quad \forall\, n \geq 1.$$

(i) *Tentukan sama ada* $\{ a_n \}$ *menokok atau menyusut.*

(ii) *Adakah* $\{ a_n \}$ *terbatas dari atas ?*

(iii) *Adakah* $\lim\limits_{n\to\infty} a_n$ *wujud ? Jika ya, cari had ini.*

(iv) *Adakah siri* $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n^2}$ *menumpu atau mencapah ?*

(Berikan alasan untuk menyokong jawapan anda)

(c) *Jika had jujukan* $\{ b_n \}$ *wujud, buktikan bahawa had ini adalah unik.*

[100 markah]

2. (a) If $z = \ell n(e^x + e^y)$, show that

(i) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$

(ii) $\dfrac{\partial^2 z}{\partial x^2} \cdot \dfrac{\partial^2 z}{\partial y^2} - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0.$

(b) (i) Show that the Maclaurin series for the function $f(x) = \dfrac{1}{1-x}$ is

$$\sum_{k=0}^{\infty} x^k$$

and find its interval of convergence.

(ii) Hence find the series for $\dfrac{1}{(1-2x)^2}$ and state its interval of convergence.

(iii) Find the sum of the series

$$1 + \frac{2\cdot 2}{5} + \frac{2^2\cdot 3}{5^2} + \frac{2^3\cdot 4}{5^3} + \cdots$$

(c) Let $\phi(u, v, w)$ be a differentiable function and u, v, w be related to x, y, z by

$$u = \frac{x}{z}, \quad v = \frac{y}{z} \quad \text{and} \quad w = z.$$

(i) Show that the equation $x\dfrac{\partial \phi}{\partial x} + y\dfrac{\partial \phi}{\partial y} + z\dfrac{\partial \phi}{\partial z} = \phi$ can be transformed into

$$w\frac{\partial\phi}{\partial w}=\phi .$$

(ii) Show that $\dfrac{\partial^2\phi}{\partial y\,\partial x}=\dfrac{1}{w^2}\dfrac{\partial^2\phi}{\partial v\,\partial u}$.

[100 marks]

2. *(a)* *Jika* $z=\ell n(e^x+e^y)$, *tunjukkan bahawa*

(i) $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=1$

(ii) $\dfrac{\partial^2 z}{\partial x^2}\cdot\dfrac{\partial^2 z}{\partial y^2}-\left(\dfrac{\partial^2 z}{\partial x\,\partial y}\right)^2=0.$

(b) *(i)* *Tunjukkan bahawa siri Maclaurin bagi* $f(x)=\dfrac{1}{1-x}$ *ialah*

$$\sum_{k=0}^{\infty} x^k$$

dan cari selang penumpuannya.

(ii) *Dengan ini cari siri bagi* $\dfrac{1}{(1-2x)^2}$ *dan nyatakan selang penumpuannya.*

(iii) *Cari hasil tambah siri*

$$1+\frac{2\cdot 2}{5}+\frac{2^2\cdot 3}{5^2}+\frac{2^3\cdot 4}{5^3}+\cdots$$

(c) *Andaikan* $\phi(u,v,w)$ *adalah suatu fungsi yang terbezakan dan* u, v, w *berhubung dengan* x, y, z *secara*

$$u=\frac{x}{z},\quad v=\frac{y}{z}\quad \text{and}\quad w=z.$$

(i) *Tunjukkan bahawa persamaan* $x\dfrac{\partial\phi}{\partial x}+y\dfrac{\partial\phi}{\partial y}+z\dfrac{\partial\phi}{\partial z}=\phi$ *dapat dijelmakan kepada*

$$w\frac{\partial\phi}{\partial w}=\phi .$$

(ii) *Tunjukkan bahawa* $\dfrac{\partial^2\phi}{\partial y\,\partial x}=\dfrac{1}{w^2}\dfrac{\partial^2\phi}{\partial v\,\partial u}$.

[100 markah]

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3 (a) Given that $f(x, y) = y^3 + 3x^2 + 6xy - 9y, \quad (x, y) \in \mathbb{R}^2$.
Classify the critical points of f and find the local extremum of f.

(b) Given $f(x,y) = \begin{cases} \dfrac{4x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

(i) Show that f is continuous at $(0, 0)$.

(ii) Find $\dfrac{\partial f}{\partial x}(x, y)$ at $(x, y) \neq (0, 0)$ and $\dfrac{\partial f}{\partial x}(0, 0)$.

(iii) Determine whether $\dfrac{\partial f}{\partial x}$ is continuous at $(0, 0)$?

(c) Determine whether each of the following series converges or diverges.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{4k+1}{\sqrt{k}\,(2k+5)}$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{4^k + 7k}{k!}$

(iii) $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{k^2+k} - k\right)$

(iv) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ell n\, k)^{\ell n\, k}}$ (Hint: $u^2 = e^{2\ell n u}$)

[100 marks]

3 *(a)* *Diberi* $f(x, y) = y^3 + 3x^2 + 6xy - 9y, \quad (x, y) \in \mathbb{R}^2$.
Kelaskan titik genting bagi f *dan cari ekstremum setempat bagi* f.

(b) *Diberi* $f(x,y) = \begin{cases} \dfrac{4x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

(i) *Tunjukkan bahawa* f *adalah selanjar pada* $(0, 0)$.

(ii) *Cari* $\dfrac{\partial f}{\partial x}(x, y)$ *pada* $(x, y) \neq (0, 0)$ *dan* $\dfrac{\partial f}{\partial x}(0, 0)$.

(iii) *Tentukan sama ada* $\dfrac{\partial f}{\partial x}$ *adalah selanjar pada* $(0, 0)$ *?*

(c) *Tentukan sama ada setiap siri yang berikut adalah menumpu atau mencapah.*

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{4k+1}{\sqrt{k}\,(2k+5)}$

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(ii) $\sum_{k=1}^{\infty} \frac{4^k + 7k}{k!}$

(iii) $\sum_{k=1}^{\infty} \left(\sqrt{k^2+k} - k\right)$

(iv) $\sum_{k=2}^{\infty} \frac{1}{(\ell n k)^{\ell n k}}$ *(Hint:* $u^2 = e^{2\ell n u}$*)*

[100 markah]

4 (a) Determine whether each of the following improper integrals converges.

(i) $\int_1^{\infty} \frac{x \cos^2 x}{\sqrt{2x^5 + 3}}\, dx$

(ii) $\int_{3^+}^{7} \frac{1}{(x^2 - x - 6)^{\frac{3}{2}}}\, dx$

(b) Evaluate the integrals :

(i) $\iint_A (3 + 2x)\; dx\, dy$

where A is the region bounded by $y = x^2$ and $y = 4 - x^2$

(ii) $\int_0^{\frac{1}{2}} \int_{\sqrt{2y}}^{1} \frac{x^2}{\sqrt{1+3x^5}}\, dx\; dy$

(iii) $\iint_D \sin \pi (x^2 + y^2)\; dx\, dy$

where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

(c) Find the volume of the solid which is bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the xy–plane, and that lies outside the cylinder $x^2 + y^2 = 1$.

[100 marks]

4 (a) Tentukan sama ada kamiran tak wajar berikut adalah menumpu.

(i) $\int_1^{\infty} \frac{x \cos^2 x}{\sqrt{2x^5 + 3}}\, dx$

(ii) $\int_{3^+}^{7} \frac{1}{(x^2 - x - 6)^{\frac{3}{2}}}\, dx$

(b) *Nilaikan kamiran berikut :*

(i) $\iint_A (3+2x)\ dx\,dy$

mana A *adalah kawasan yang dibatasi oleh* $y = x^2$ *dan* $y = 4 - x^2$

(ii) $\int_0^{\frac{1}{2}} \int_{\sqrt{2y}}^{1} \frac{x^2}{\sqrt{1+3x^5}}\, dx\ dy$

(iii) $\iint_D \sin \pi (x^2 + y^2)\ dx\,dy$

di mana $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.

(c) *Cari isipadu bongkah yang dibatasi di atas oleh paraboloid* $z = 9 - x^2 - y^2$ *dan di bawah oleh satah* xy, *dan terletak di luar silinder* $x^2 + y^2 = 1$.

[100 markah]

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