UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2006/2007

April 2007

MAT 102 – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

[Arahan: *Jawab* ***semua empat*** *[4] soalan.]*

1. (a) Find the following limits of sequences or functions:

(i) $\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$ (ii) $\lim_{n\to\infty}\frac{6n^3+\cos^2 n}{2n^4+n-1}$

(iii) $\lim_{x\to 1}\left(\frac{1}{\ell n x}-\frac{x}{x-1}\right)$ (iv) $\lim_{x\to 0^+}\frac{\sin x\ e^{-\frac{1}{x}}}{x^2}$

(n is a positive integer and x is a real number)

(b) The sequence $\{a_n\}$ is defined as $a_1=5$, $a_{n+1}=8-\frac{7}{a_n}$, $\forall\, n\geq 1$.

(i) Show that $a_n\leq 7$, $\forall\, n\geq 1$.

(ii) Is $\{a_n\}$ bounded below?

(iii) Show that $\{a_n\}$ is an increasing sequence.

(iv) Does $\lim_{n\to\infty} a_n$ exist? Give reasons to support your answer. If yes, find this limit.

(c) (i) If $\sum_{k=1}^{\infty}|a_k|$ converges, prove that $\sum_{k=1}^{\infty} a_k$ also converges.

(ii) Give an example to show that the converse of (i) is not true, i.e. give a series $\sum_{k=1}^{\infty} a_k$ which converges but $\sum_{k=1}^{\infty}|a_k|$ diverges.

[100 marks]

2. (a) Given $f(x,y)=\begin{cases}\frac{x^2-xy^2-y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

(i) Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists. Is f continuous at $(0,0)$?

(ii) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

.../3-

1. (a) *Cari had jujukan atau fungsi yang berikut:*

(i) $\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$ (ii) $\lim_{n\to\infty}\frac{6n^3+\cos^2 n}{2n^4+n-1}$

(iii) $\lim_{x\to 1}\left(\frac{1}{\ell n x}-\frac{x}{x-1}\right)$ (iv) $\lim_{x\to 0^+}\frac{\sin x\ e^{-\frac{1}{x}}}{x^2}$

(n ialah integer positif dan x ialah number nyata)

(b) *Jujukan* $\{a_n\}$ *ditakrifkan seperti berikut:*

$$a_1=5,\quad a_{n+1}=8-\frac{7}{a_n},\quad \forall\, n\geq 1.$$

(i) *Tunjukkan bahawa* $a_n\leq 7,\quad \forall\, n\geq 1.$

(ii) *Adakah* $\{a_n\}$ *terbatas dari bawah?*

(iii) *Tunjukkan bahawa jujukan* $\{a_n\}$ *adalah menokok.*

(iv) *Wujudkah* $\lim_{n\to\infty} a_n$*? Berikan alasan untuk menyokong jawapan anda. Jika ya, cari had ini.*

(c) (i) *Jika* $\sum_{k=1}^{\infty}|a_k|$ *menumpu, buktikan bahawa* $\sum_{k=1}^{\infty}a_k$ *menumpu juga.*

(ii) *Berikan suatu contoh untuk menunjukkan bahawa akas penyataan dalam bahagian (i) adalah tidak benar, iaitu berikan suatu siri* $\sum_{k=1}^{\infty}a_k$ *yang menumpu tetapi* $\sum_{k=1}^{\infty}|a_k|$ *mencapah.*

[100 markah]

2. (a) *Diberi* $f(x,y)=\begin{cases}\dfrac{x^2-xy^2-y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

(i) *Tentukan sama ada* $\lim_{(x,y)\to(0,0)} f(x,y)$ *wujud.*
Adakah f *selanjar pada* $(0,0)$*?*

(ii) *Cari* $\frac{\partial f}{\partial x}(0,0)$ *dan* $\frac{\partial f}{\partial y}(0,0)$.

.../4-

(b) (i) Suppose $w = \ell n(ax+by)+\tan(ax-by)$, where a and b are non-zero constants. Show that

$$b^2\frac{\partial^2 w}{\partial x^2} = a^2\frac{\partial^2 w}{\partial y^2}.$$

(ii) Given that $z = x^3 e^{\frac{y}{x}}$, $x = u-\cos v$ and $y = \sin(2u+v^2)$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

(c) (i) Show that the Maclaurin series for the function $f(x) = e^x$ is

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

and find its interval of convergence.

(ii) By using (i), find the Maclaurin series for e^{x^2} and its interval of convergence.

(iii) Show that

$$\int_0^1 x e^{x^2}dx = \sum_{k=0}^{\infty}\frac{1}{k!(2k+2)}.$$

(iv) Hence find the sum of the series

$$\sum_{k=0}^{\infty}\frac{1}{k!(2k+2)}.$$

[100 marks]

3. (a) Determine whether each of the following series converges or diverges.

(i) $\sum_{k=1}^{\infty}\frac{k!}{7^k(k+3)}$ (ii) $\sum_{k=2}^{\infty}\frac{k^2+1}{k\,\ell n\,k}$ (iii) $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$

(iv) $\sum_{k=1}^{\infty}a_k$ where $\sum_{k=1}^{\infty}a_k$ is a positive term series and $\lim_{k\to\infty} ka_k = \alpha > 0$.

(b) The function f is twice differentiable and $u(x,t) = f(x^2-t)$. Show that

$$\frac{\partial^2 u}{\partial x^2}-4x^2\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t} = 0.$$

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(b) (i) *Andaikan* $w = \ell n(ax+by) + \tan(ax-by)$, *di mana* a *dan* b *adalah pemalar yang bukan sifar. Tunjukkan bahawa*

$$b^2 \frac{\partial^2 w}{\partial x^2} = a^2 \frac{\partial^2 w}{\partial y^2}.$$

(ii) *Diberi* $z = x^3 e^{\frac{y}{x}}$, $x = u - \cos v$ *dan* $y = \sin(2u+v^2)$.
Cari $\frac{\partial z}{\partial u}$ *and* $\frac{\partial z}{\partial v}$.

(c) (i) *Tunjukkan bahawa siri Maclaurin bagi fungsi* $f(x) = e^x$ *ialah*

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

dan cari selang penumpuannya.

(ii) *Dengan menggunakan (i), cari siri Maclaurin bagi* e^{x^2} *dan selang penumpuannya.*

(iii) *Tunjukkan bahawa*

$$\int_0^1 x e^{x^2} dx = \sum_{k=0}^{\infty} \frac{1}{k!(2k+2)}.$$

(iv) *Dengan ini, cari jumlah siri*

$$\sum_{k=0}^{\infty} \frac{1}{k!(2k+2)}.$$

[100 markah]

3. (a) *Tentukan sama ada setiap siri yang berikut menumpu atau mencapah.*

(i) $\sum_{k=1}^{\infty} \frac{k!}{7^k(k+3)}$ (ii) $\sum_{k=2}^{\infty} \frac{k^2+1}{k\,\ell n\,k}$ (iii) $\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$

(iv) $\sum_{k=1}^{\infty} a_k$ *di mana* $\sum_{k=1}^{\infty} a_k$ *ialah suatu siri dengan sebutan positif dan* $\lim_{k\to\infty} k a_k = \alpha > 0$.

(b) *Fungsi* f *mempunyai terbitan peringkat kedua dan* $u(x,t) = f(x^2 - t)$. *Tunjukkan bahawa*

$$\frac{\partial^2 u}{\partial x^2} - 4x^2 \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 0.$$

.../6-

(c) The temperature T on an elliptical plate $D=\left\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{2}+\frac{y^2}{16}\le 1\right\}$ is given by $T(x,y)=8x^2+24xy+y^2$.

(i) Find the hottest point(s) on D and the highest temperature.

(ii) At the point $\left(\frac{1}{2},\frac{1}{2}\right)$, what is the rate of increase in temperature in the direction $(1,\ 0)$?

(iii) What is the direction at $\left(\frac{1}{2},\frac{1}{2}\right)$ that gives the maximum rate of increase in temperature?

[100 marks]

4. (a) Determine whether each of the following improper integrals converges.

(i) $\int_1^{\infty} \frac{2x}{e^x(x^2+3)}\,dx$

(ii) $\int_{0^+}^{1} \frac{\sin x}{x\sqrt{x}}\,dx$

(b) Evaluate the integrals :

(i) $\int_0^4 \int_x^4 \frac{3\cos 2y}{y}\,dy\,dx$

(ii) $\iint_K (x^2+y^2)^{\frac{1}{2}} \tan^{-1}\frac{y}{x}\,dxdy$

where K is the region in the first and second quadrants which is bounded by $y=x$, $y=-x$ and $x^2+y^2=9$.

(c) Let V be the pyramid bounded by the coordinate planes $x=0$, $y=0$, $z=0$ and the plane $x+y+2z=2$.

(i) Sketch the solid V and find the volume of V by using double integral.

(ii) Find the triple integral $\iiint_V e^{x+2y+3z}\,dxdydz$.

[100 marks]

.../7-

(c) *Suhu* T *pada suatu plat eliptik* $D=\left\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{2}+\frac{y^2}{16}\le 1\right\}$ *diberi dengan* $T(x,y)=8x^2+24xy+y^2$.

(i) *Cari titik(-titik) pada* D *yang terpanas dan suhu yang tertinggi.*

(ii) *Pada titik* $\left(\frac{1}{2},\frac{1}{2}\right)$, *apakah kadar peningkatan suhu dalam arah* $(1,\ 0)$*?*

(iii) *Dalam arah apakah pada titik* $\left(\frac{1}{2},\frac{1}{2}\right)$ *yang memberikan kadar peningkatan maksimum dalam suhu?*

[100 markah]

4. (a) *Tentukan sama ada setiap kamiran tak wajar yang berikut adalah menumpu.*

(i) $\int_1^{\infty} \frac{2x}{e^x(x^2+3)}\,dx$

(ii) $\int_{0^+}^{1} \frac{\sin x}{x\sqrt{x}}\,dx$

(b) *Nilaikan kamiran yang berikut :*

(i) $\int_0^4 \int_x^4 \frac{3\cos 2y}{y}\,dy\,dx$

(ii) $\iint_K (x^2+y^2)^{\frac{1}{2}} \tan^{-1}\frac{y}{x}\,dxdy$

di mana K *ialah rantau dalam sukuan pertama dan kedua yang dibatasi oleh* $y=x$, $y=-x$ *dan* $x^2+y^2=9$.

(c) *Andaikan* V *piramid yang dibatasi oleh satah koordinat* $x=0$, $y=0$, $z=0$ *dan satah* $x+y+2z=2$.

(i) *Lakarkan* V *dan cari isipadu* V *dengan menggunakan kamiran berganda dua.*

(ii) *Cari kamiran berganda tiga* $\iiint_V e^{x+2y+3z}\,dxdydz$.

[100 markah]

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