UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2005/2006

November 2005

MAT 101E – Calculus
[Kalkulus]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consist of **SIX** pages of printed material before you begin the examination.

Sila pastikan bahawa kertas peperiksaan ini mengandungi ***ENAM*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

Instructions: Answer **all four** [4] questions.

[Arahan: Jawab ***semua empat*** *soalan].*

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1. (a) Prove by using definition that $\lim_{x \to 2} \frac{3x^2}{x+1} = 4$.

(b) Find the following limits :

(i) $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$

(ii) $\lim_{x \to 4} \frac{x-4}{3(\sqrt{x}-2)}$

(iii) $\lim_{x \to 3^-} 3x^2 + 2\left[\frac{x}{2} \right] - \frac{|x^2-x-6|}{x-3}$,

where $[x]$ is the greatest integer function.

(iv) $\lim_{x \to a} (2x^2 - 5ax + 3a^2) \cos \frac{1}{x-a}$

(c) Prove that if f is differentiable at $x = a$, then f is also continuous at $x = a$.

[100 marks]

1. (a) Dengan menggunakan takrif had, buktikan bahawa $\lim_{x \to 2} \frac{3x^2}{x+1} = 4$.

(b) Cari had yang berikut :

(i) $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$

(ii) $\lim_{x \to 4} \frac{x-4}{3(\sqrt{x}-2)}$

(iii) $\lim_{x \to 3^-} 3x^2 + 2\left[\frac{x}{2} \right] - \frac{|x^2-x-6|}{x-3}$,

di mana $[x]$ *ialah fungsi integer terbesar.*

(iv) $\lim_{x \to a} (2x^2 - 5ax + 3a^2) \cos \frac{1}{x-a}$.

(c) Buktikan bahawa jika f *adalah terbezakan pada* $x = a$, *maka* f *juga selanjar pada* $x = a$.

[100 markah]

2 (a) Suppose that $f(x) = \begin{cases} \cos x, & x < 0 \\ x^2, & 0 \le x < 1 \\ Ax + B, & x \ge 1 \end{cases}$, where A and B are constants.

(i) Determine whether f is continuous at $x = 0$.

(ii) If f is continuous at $x = 1$, show that $A + B = 1$.

(iii) Find A and B so that f is differentiable at $x = 1$ by using the definition of derivative.
What is the value of $f'(1)$?

(b) Consider the equation $x^5 + \frac{x^3}{3} - 2x^2 + 4x = 1$.

(i) Show that this equation has a solution in $\mathbb{R}$.

(ii) Prove that its solution is unique.

(c) Suppose that $g(x) = \ell n(1 + x) - x, \ \forall\, x > 0$.

(i) Is g strictly increasing or strictly decreasing for $x > 0$. Give your reasons. .

(ii) By using the result in (i) or otherwise, show that

$$\ell n(1 + x) < x, \quad \forall\, x > 0.$$

[100 marks]

2 (a) *Andaikan* $f(x) = \begin{cases} \cos x, & x < 0 \\ x^2, & 0 \le x < 1 \\ Ax + B, & x \ge 1 \end{cases}$, *di mana* A *and* B *adalah pemalar.*

(i) *Tentukan sama ada* f *selanjar pada* $x = 0$.

(ii) *Jika* f *selanjar pada* $x = 1$, *tunjukkan bahawa* $A + B = 1$.

(iii) *Cari* A *and* B *dengan menggunakan takrif terbitan supaya* f *terbezakan pada* $x = 1$.
Apakah nilai $f'(1)$ *?*

(b) *Pertimbangkan persamaan* $x^5 + \frac{x^3}{3} - 2x^2 + 4x = 1$.

(i) *Tunjukkan bahawa persamaan ini mempunyai suatu penyelesaian dalam* $\mathbb{R}$.

(ii) *Buktikan bahawa penyelesaiannya adalah unik.*

(c) Andaikan $g(x) = \ell n(1+x) - x, \ \forall\, x > 0$.

(i) Adakah g *menokok secara tegas atau menyusut secara tegas bagi* $x > 0$ *?*
Berikan alasan.

(ii) Dengan menggunakan hasil dari (i) atau cara lain, tunjukkan bahawa

$$\ell n(1+x) < x, \quad \forall\, x > 0.$$

[100 markah]

3 (a) Given $f(x) = \dfrac{2x}{x^2+2}, \ \forall\, x \in \mathbb{R}$.

(i) Find all the asymptotes of f.

(ii) Determine where f is increasing, decreasing, convex and concave.

(iii) Find the local extrema of f and the inflection points of f.

(iv) Sketch the graph of f using the above features.

(b) The function $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$.
For $x \in [0,1]$,

$$G(x) = \int_0^x f(t)\,dt,$$

$$H(x) = \int_x^1 f(t)\,dt.$$

(i) Find $G'(x)$ and $H'(x), \ \forall\, x \in (0,1)$.

(ii) If $G(x) = H(x), \ \forall\, x \in [0,1]$, prove that $f(x) = 0, \ \forall\, x \in [0,1]$.

(c) Prove that $\sin^{-1}\dfrac{1}{5} + \cos^{-1}\dfrac{1}{5} = \dfrac{\pi}{2}$.

[100 marks]

3 (a) Diberi $f(x) = \dfrac{2x}{x^2+2}, \ \forall\, x \in \mathbb{R}$.

(i) Cari semua asimptot bagi f.

(ii) Tentukan di mana f *menokok, menyusut, cembung dan cekung.*

(iii) Cari semua ekstremum bagi f *dan titik lengkok balas bagi* f.

(iv) Lakarkan graf f *dengan menggunakan ciri atas.*

(b) Fungsi $f: \mathbb{R} \to \mathbb{R}$ *selanjar pada* $\mathbb{R}$.
Bagi $x \in [0,1]$,

$$G(x) = \int_0^x f(t)\,dt,$$

$$H(x) = \int_x^1 f(t)\,dt.$$

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(i) *Cari* $G'(x)$ *and* $H'(x)$, $\forall\, x \in (0,1)$.

(ii) *Jika* $G(x) = H(x)$, $\forall\, x \in [0,1]$, *buktikan bahawa* $f(x) = 0$, $\forall\, x \in [0,1]$.

(c) *Buktikan bahawa* $\sin^{-1}\dfrac{1}{5} + \cos^{-1}\dfrac{1}{5} = \dfrac{\pi}{2}$.

[100 markah]

4. (a) Find the following integrals :

(i) $\int (4 - 2\cos x)^3 \sin x\, dx$

(ii) $\int \dfrac{x^2 + x + 1}{x^2 + 2x + 1}\, dx$

(iii) $\int \dfrac{1}{x^3\sqrt{x^2 - 9}}\, dx$

(iv) $\int \sin(\ell n\, x)\, dx$

(b) Given that $f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 5, & x = 1 \end{cases}$.

(i) Let $P_n = \left\{ 0, \dfrac{1}{n}, \dfrac{2}{n}, \cdots, \dfrac{n-1}{n}, 1 \right\}$ be a partition of the interval $[0,1]$, where n is a positive integer. Find the lower sum $L(P_n, f)$ and the upper sum $U(P_n, f)$.

(ii) Determine whether f is integrable on $[0,1]$.

(iii) If f is integrable on $[0,1]$, what is the value of $\int_0^1 f(x)\, dx$?

[100 marks]

4. (a) Cari kamiran berikut :

(i) $\int (4 - 2\cos x)^3 \sin x\, dx$

(ii) $\int \dfrac{x^2 + x + 1}{x^2 + 2x + 1}\, dx$

(iii) $\int \dfrac{1}{x^3\sqrt{x^2 - 9}}\, dx$

(iv) $\int \sin(\ell n\, x)\, dx$

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(b) *Diberi* $f(x)=\begin{cases}2, & 0\le x<1\\ 5, & x=1\end{cases}$.

(i) *Andaikan* $P_n=\left\{0,\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n},1\right\}$ *ialah suatu partisi bagi* $[0,1]$, *di mana* n *ialah suatu integer positif. Cari hasil tambah bawah* $L(P_n,f)$ *dan hasil tambah atas* $U(P_n,f)$.

(ii) *Tentukan sama ada* f *terkamirkan pada* $[0,1]$.

(iii) *Jika* f *terkamirkan pada* $[0,1]$, *apakah nilai* $\int_0^1 f(x)\,dx$?

[100 markah]

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