S-PERMUTABILITY, SEMIPERMUTABILITY, c-NORMALITY AND c-PERMUTABILITY OF SUBGROUPS IN FINITE GROUPS

by

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Thesis submitted in fulfillment of the requirements

for the degree of

Doctor of Philosophy

August 2014

ACKNOWLEDGEMENTS

I appreciate the support of many people who have contributed to this work. I would like to sincerely thank my supervisor, Dr. Hajar Sulaiman for her help and support. Her comments and suggestion were very helpful in completing this work. Her trust in me and my ability as a graduate student is greatly appreciated. And a great many thanks to Professor Dr. Khaled Ahmed Al-Sharo, for this work will not have been completed without his help, his comments and his big support.

I would like also to acknowledge all the faculty members of the School of Mathematical Sciences, at Universiti Sains Malaysia for their support and inspiration.

Finally, I would like to thank my husband Mutasem Al-Khaswneh who gave my life a special meaning and had encouraged me throughout my study. This thesis would not have been possible without his support. Needless to mention my daughters Arwa and my new baby Bushra who were a great help in keeping me awake many nights studying.

I would like to thank my father, mother, for their help, support, patience and "duaa" for me.

Lastly, I would like to thank my brothers; Dr.Mohamad, Eng.Ahmad, sisters; Dr.Ayat, Eng.Alaa, Eng.Esraa, Wesam, and my extended family for their encouragement during my study.

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List Of Symbols and Notations

$\neq$	not equal to
$\forall$	for all
$\exists$	there exist
$\Longleftrightarrow$	If and only if
$\Rightarrow$	implication
$\in$	belong to
$\notin$	not belong to
$\subset$	proper subset
$\subseteq$	subset of
$\simeq$	is isomorphic to
G	group G
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\langle H, K \rangle$	the join of subgroups H, K of G
$\langle g \rangle$	the cyclic group generated by g
p	prime number.
$\sigma(G)$	the set of prime divisors of the group G
mod	modulo
$\equiv$	congruent

$a \equiv b (\mathrm{mod}\ n)$	a congruent b modulo n
$p \nmid m$	the prime number p does not divide m
$Aut(G)$	the set of all automorphisms of a group G
Z_n	the cyclic group of order n
V_4	the Klein 4-group
D_{2n}	the dihedral group of order $2n$
S_n	the symmetric group of a set with n letters
$\phi : X \longrightarrow Y$	ϕ is a map of set X into a set Y
$A \subseteq B$	A is a subset of B
G/H	the factor group or quotient of G by H
$(G : H)$	the index of the subgroup H in the group G
$\|G\|$	the order of G
HK	the set $\{hk \| h \in H, k \in K\}$
$C_G(H)$	the centralizer of H in G
$Z(G)$	the center of the group G
$N_G(H)$	the normalizer of the group G
G'	the commutator subgroup of G
$Fit(G)$	the fitting subgroup of G
$Fit^*(G)$	the generalized fitting subgroup of G
$\phi(G)$	frattini subgroup
H^x	the conjugate of H by x

$[a, b]$	the commutator of the elements a and b
$Syl_p(G)$	the set of Sylow p-subgroup of G
$H \lhd\lhd G$	H is a subnormal subgroup in G
$H \lhd .G$	H is a minimal normal subgroup in G
$H\ Char\ G$	H is characteristic subgroup of G
$G^{\mathfrak{X}}$	$\mathfrak{X}$-residual of G
H_G	the core of H in G
H_{PG}	the P-core of H in G
H c-norm G	H is c-normal in G
H c-perm G	H is c-permutable in G
$O_p(G)$	the *p-core* of a finite group
$O_{p\prime}(G)$	the *p′-core* of a finite group
(g, h)	the greatest common divisor
T-group	a group in which the property of normality is transitive relation
PT-group	a group in which the property of permutability is transitive relation
PST-group	a group in which the property Sylow permutability is transitive relation
BT-group	a group in which the property semipermutability is transitive relation
BST-group	a group in which the property S-semipermutability is transitive

relation

SN-group a group in which every subnormal subgroup is seminormal

CT-group a group in which the property of c-normality is transitive relation

CPT-group a group in which the property of c-permutability is transitive relation

SC_N-group a group with all its all subnormal subgroups are c-normal

SC_P-group a group with all its subnormal subgroups are c-permutable

KETERPILIHATURAN-S, KESEMITERPILIHATURAN, KENORMALAN-c, DAN KETERPILIHATURAN-c KE ATAS SUBKUMPULAN DALAM KUMPULAN TERHINGGA

ABSTRAK

Semua kumpulan dan subkumpulan yang dipertimbangkan dalam tesis ini adalah terhingga. Dalam tesis ini, beberapa teori baru dan bukti berkaitan terhadap kumpulan terpilihatur-S (juga dirujuk sebagai terpilihatur-Sylow), semiterpilihatur, normal-c dan terpilihatur-c dari suatu kumpulan terhingga G dibincangkan. Lebih daripada satu topik difokuskan di sini, antaranya adalah, hubungan di antara kelas subkumpulan terpilihatur-S terbenam dan kumpulan-PST, subkumpulan semiterpilihatur, dan keadaan di mana subkumpulan normal-c dan subkumpulan terpilihatur-c adalah transitif. Penemuan baru mendedahkan bahawa kumpulan-PST, iaitu kumpulan yang mana keterpilihaturan-S adalah suatu hubungan transitif, bertindih dengan kelas kumpulan terselesaikan terhingga yang mana setiap subkumpulan-p adalah terpilihatur-S terbenam. Beberapa fakta telah dibangunkan untuk membuktikan hasil ini, khususnya, suatu bukti termudah diberi untuk membuktikan bahawa jika suatu subkumpulan adalah terpilihatur-S terbenam dalam suatu kumpulan maka ia mestilah terpilihatur-S terbenam dalam semua subkumpulan yang mengandunginya. Suatu hasil pendua juga telah ditunjukkan untuk kumpulan yang dipanggil sebagai kumpulan-PST_c yang mana "c" menandakan syarat istimewa terhadap sisa nilpoten. Penyelidikan ke atas kesemiterpilihaturan subkumpulan dan hubungan terhadap ketransitifan subkumpulan tersebut telah diterokai secara meluas dalam beberapa tahun yang lepas. Apabila kesemiterpilihaturan dalam kumpulan G adalah suatu hubungan transitif, G dipanggil sebagai kumpulan-BT. Apabila semua kumpulan subnormal

dari G adalah semiterpilihatur, G dipanggil sebagai kumpulan-SP. Telah pun ditunjukkan bahawa semua kumpulan-BT adalah kumpulan-SP. Walau bagaimanapun, suatu kumpulan-SP tidak semestinya suatu kumpulan-BT sebagaimana yang ditunjukkan dalam tesis ini. Bagi keadaan di mana kenormalan-c adalah suatu hubungan transitif, suatu kelas kumpulan baru yang dipanggil kumpulan-CT diperkenalkan dan beberapa sifat serta teorem yang berkaitan dengannya dibuktikan. Beberapa keputusan baru berkaitan dengan kumpulan-CPT, iaitu kumpulan yang mana keterpilihaturan-c adalah suatu hubungan transitif, juga diberi. Tesis ini juga mengandungi bukti bahawa wujud suatu subkumpulan terpilihatur-c yang bukan normal-c. Perincian baru terhadap kumpulan dan subkumpulan berikut ditakrifkan dan dibuktikan di dalam tesis ini iaitu: kumpulan-PST terhingga terselesaikan, subkumpulan terpilihatur-S terbenam, kumpulan-BT, subkumpulan semiterpilihatur, subkumpulan seminormal, kumpulan-CT terhingga terselesaikan, kumpulan-bukan-CT minimum, kumpulan-bukan-CPT minimum, kumpulan yang mana semua subkumpulan subnormalnya adalah normal-c (dipanggil kumpulan-SC_N) dan kumpulan yang mana semua subkumpulan subnormalnya adalah terpilihatur-c (dipanggil kumpulan-SC_P).

S-PERMUTABILITY, SEMIPERMUTABILITY, c-NORMALITY AND c-PERMUTABILITY OF SUBGROUPS IN FINITE GROUPS

ABSTRACT

All groups and subgroups considered in this thesis are finite. In this thesis, some new theories and related proofs on S-permutable (also referred to as Sylow permutable), semipermutable, c-normal and c-permutable subgroups of a finite group G are discussed. More than one topic are focused here, some of which are, the relationship between classes of S-permutable embedded subgroups and PST-groups, the semipermutable subgroups, and the condition when c-normal and c-permutable subgroups are transitive. New findings reveal that PST-groups, the class of groups in which S-permutability is a transitive relation, coincides with the class of finite solvable groups in which every p-subgroup is S-permutably embedded. Some facts were developed to prove this result, in particular, a simplified proof is given to prove that if a subgroup is S-permutably embedded in the group then it should be S-permutably embedded in every subgroup that contains it. A dual result is also established for a group called the PST_c-group where "c" denotes the special condition on the nilpotent residual. Research on semipermutability of subgroups and their transitivity relationships have been extensively explored in previous years. When semipermutability in a group G is a transitive relation, G is called a BT-group. When all subnormal subgroups of G are semipermutable, G is called an SP-group. It has been shown that all BT-groups are SP-groups. However, an SP-group is not necessarily a BT-group as shown in the thesis. For the condition in which c-normality is a transitive relation, a new class of group called CT-groups is introduced and some related properties and theorems to this group are proved. Some new results related to CPT-groups, which are groups in which c-

permutability is a transitive relation, are also given. This thesis also contains the proof that there exists a c-permutable subgroup which is not c-normal. New characterizations of the following groups and subgroups are defined and proved in this thesis which are: the finite solvable PST-groups, S-permutably embedded subgroups, BT-groups, semipermutable subgroups, seminormal subgroups, finite solvable CT-groups, minimal non-CT-groups, minimal non-CPT-groups and groups with all its subnormal subgroups c-normal (called SC_N–groups), and groups with all its subnormal subgroups c-permutable (called SC_p-groups).

Chapter 1

INTRODUCTION

1.1 Introduction

This chapter gives an introduction to the study. It begins with some related background theory and literature review which provides a comprehensive background knowledge on permutable and semipermutable subgroups, the property of c-normality and c-permutability of subgroups, as well as transitivity of such properties in relation to a finite group G. This is followed by the problem statement, the objectives as well as the scope of the study, the research methodology and lastly the thesis organization. All groups and subgroups described in this thesis are finite.

1.2 Background and Literature Review

This study consists of investigations on several concepts of subgroup properties in a finite group G. One concept is permutability (which converts to S-permutability for Sylow subgroups), semipermutability (which converts to S-semipermutability for Sylow subgroups), c-permutability and c-normality. Then, to each of these properties, there exists what is called a transitivity property where there are groups defined and given special names when a subgroup property, for example permutability, is a transitive relation in G. These properties, as well as the related transitivity, will be described briefly in what follows.

The product HK of two subgroups H and K of a group G is not always a subgroup of G. For example, if $G = S_3$, then consider $H = \langle(12)\rangle$ and $K = \langle(13)\rangle$, two subgroups of G. Then the product $HK = \{(123), (12), (13), e\}$ is not a subgroup of G. In fact, HK is a subgroup if and only if $HK = KH$ and if this is the case then H is said to permute with K. A permutable subgroup is a subgroup H of G that permutes with every subgroup of G. A subgroup H of a group G is subnormal if there is a finite chain of subgroups, each one normal in the next, beginning at H and ending at G. Subnormality is also a concept that will be discussed in this thesis.

The notion of permutable subgroups was introduced when it was observed that there are subgroups which are not normal but still commute with every subgroup of a finite group G. Every normal subgroup is permutable, but the converse is not true. In fact, there are groups in which every subgroup is permutable, but none were normal. Permutable subgroups were initially termed quasinormal subgroups in 1939 by Ore [23]. The study of permutable subgroups has resulted to many interesting properties especially when G is a finite group. It was observed that every permutable subgroup H of a finite group G is subnormal [23]. If a subgroup G permutes with every Sylow subgroup of G, then it is called an S-permutable (or Sylow-permutable) subgroup. According to Agrawal[1], Kegel introduced the concept of S-permutability in 1962 and proved that an S-permutable subgroup is always subnormal . Deskins [15] further extended this concept in 1963 and explained that S-permutable subgroups have similar properties to permutable subgroups.

From permutability, a special condition is set to obtain subgroups which are semipermutable. A subgroup H in G is semipermutable if it permutes with every subgroup K of G with $(|K|, |H|) = 1$. Similar to the permutable concept, this can be extended to S-semipermutability when Sylow subgroups are involved. Permutable (resp. S-permutable) subgroups are semipermutable (resp. S-semipermutable) but the converse is not true. In 2005, Zhang and Wang [36] used S-semipermutable subgroups of prime power to determine the structure of finite groups [1].

Permutability is a transitive relation in a group G if for any two subgroups H and K of G such that H is permutable in K and K is permutable in G then it implies that H is permutable in G. In this case, G is called a PT-group. In 1964, Zacher, defined and studied PT-groups. He also classified solvable PT-groups and proved that these are groups with a normal abelian Hall subgroup L of G with odd order and G/L is an Iwasawa group. He also proved that a solvable PT-group is supersolvable. When semipermutability is a transitive relation in G (similarly, transitivity here means that for any two subgroups H and K of G such that H is semipermutable in K and K is semipermutable in G then it implies that H is semipermutable in G), then G is called a BT-group. This was introduced by Wang et al. in [32], where they also classified solvable BT-groups. Other characterizations of solvable BT-groups were established in 2010 by Al-Sharo et al. [3]. In this study, BT-groups are investigated in relation to SP-groups. SP-groups is a special name given to a class of groups in which all subnormal subgroups are semipermutable. From the definition, it is clear that a BT-group is an SP-group. In this thesis, it shall be shown that the converse is not true.

In 1975, Agrawal [1] defined PST-groups. These are groups in which S-

permutability is a transitive relation (from now on, transitivity is presumed to be well-understood). According to Agrawal, PST-groups are exactly those groups in which all subnormal subgroups are S-permutable. In addition, he proved that if G_1 and G_2 are two PST-groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is also a PST-group. Furthermore, Agrawal [1] showed that these are exactly the groups with a normal abelian Hall subgroup L of G with odd order; where G/L is nilpotent group; and the element of G acts by conjugation as power automorphism on L. He also proved that a solvable PST-group is supersolvable. In this study, the link between the solvability of PST-groups and the S-permutably embedded subgroups are established. The concept of subgroup embedding will be described in the following chapters.

The concept of c-normality was introduced in 1996 by Wang [32], where a subgroup H is defined to be c-normal in a group G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$.Wang used the concept of c-normality to prove some known theorems by replacing normal subgroups with c-normal subgroups. Furthermore, he used c-normality in maximal subgroups to give some conditions for the solvability and supersolvability. Now, groups in which c-normality are a transitive relation have not been established. In this study, this class of groups are defined and some facts and properties related to these groups are stated and proved.

c-Permutability is a concept that was defined and established by Al-Sharo and Sulieman [4]. A subgroup H is c-permutable in a group G if there exists a permutable subgroup P of G such that $HP = G$ and $H \cap P \leq H_{PG}$. Al-Sharo and Sulieman also described transitivity of this property where they named the class of groups in which c-permutability is transitive as CPT-groups. Moreover, they proved new facts about CPT-groups on supersolvability and direct product and also described this group as a normal abelian Hall subgroup L of odd order. They also proved that if G is a CPT-group then every subgroup of the group G/L is c-normal and that G acts by conjugation as a power automorphism on L. In this study, some further properties of CPT-groups are stated and proven. The link between c-normality and c-permutability is also mentioned in this thesis.

Related to all these classes of groups, it is possible to define classes of groups that do not satisfy the transitive relation but every proper subgroup it contains does. For example, *non-*PST-groups are groups in which S-permutability is not transitive but all proper subgroups in each group are PST-groups. The minimal *non-*PST and *non-*PT groups were classified by Robinson in [26]. Later, Wang et al. [32] gave the structure of the minimal *non-*BT-groups (resp. *non-*SBT-groups, where an SBT-group is a group in which S-semipermutability is a transitive relation). In this thesis, two minimal non-X-groups are established (for the properties of X when c-normality is transitive and when c-permutability is transitive) and some facts on these groups are stated and proved.

Recall that SP-groups are groups in which all subnormal subgroups are semipermutable. If all subnormal subgroups of a group G is S-semipermutable, then G is called an SPS-group instead. Beidleman and Ragland [**11**] studied this class of groups and showed that if G is a solvable group, then G is a PST-group if and only if G is an SPS-group. As with SP-groups and SPS-groups, other groups can be established with regards to subnormal subgroups. In this thesis, two new classes of groups are established; one in which all subnormal subgroups are c-normal and the other is in which all subnormal subgroups are c-permutable.

Other than the known groups already mentioned, there are also T-groups, groups in which normality is a transitive relation. Best and Taussaky [13] defined and studied T-groups. In 1957, Gaschutz described solvable T-groups as groups in which nilpotent residual is an abelian Hall subgroup L of odd order such that G/L is a Dedekind group and G normalizes L [1]. It was also proved that a solvable T-group is supersolvable. Since normality is not a concept that is concentrated in this study, then this class of group will only be mentioned in Chapter 2, in the preliminaries. Other concepts such as the X_p-groups, seminormal subgroups and the C_p-condition will be explained in Chapter 2 also and in the following chapters where these groups are discussed.

1.3 Problem Statement

Several classes of groups have been found in recent years relating to the permutability (resp. S-permutability), semipermutability (resp. S-semipermutability) and c-permutability. However, not all properties related to the groups have been completely explored. Further studies need to be done on the existing groups to be able to state new theorems and establish new links. In addition, new classes of groups that have not been found must be established and new related facts or theorems must be stated and proved.

1.4 Scope and Objectives of the Study

The scope of this research is to prove further facts and theories related to the permutability and semipermutability, as well as c-normality and c-permutability of subgroups in a finite group G. This study also focuses in defining new classes of groups and state and prove new theorems on these groups.

The following are the objectives of this study:

1. To discover some properties of Sylow p-subgroups and S-permutable subgroups.

2. To link between S-permutably embedded subgroups and the solvability of PST-groups with and without a special condition on nilpotent residual.

3. To show that Dihedral groups are always BT-groups and BT-groups it is not equivalent to SP-groups.

4. To define a new class of groups in which c-normality is transitive and for the solvable case, prove some facts and theorems in relation to supersolvability, direct product, nilpotent residual and other properties.

5. To show that c-permutable subgroups are not equivalent to c-normal subgroups.

6. To find and prove new theorems on CPT-groups.

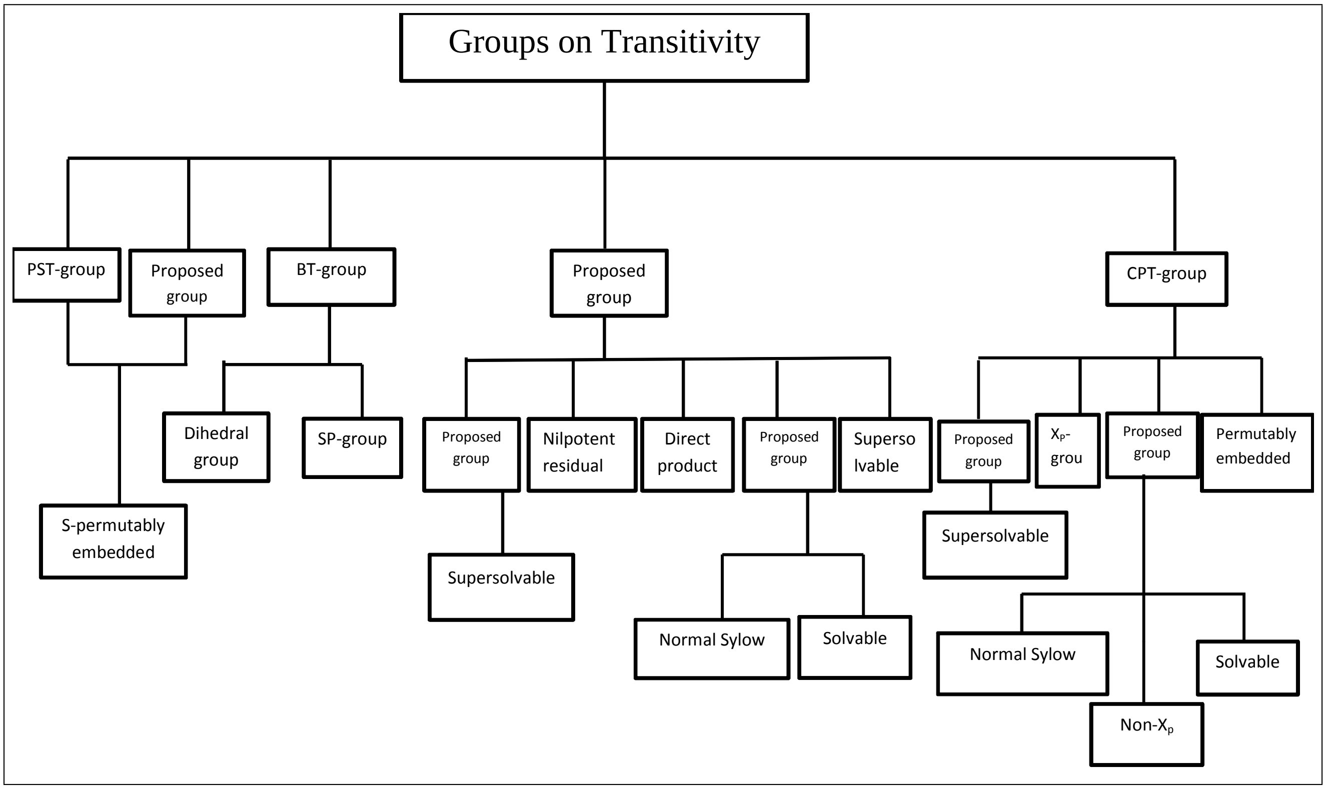
7. To establish two new groups: one in which all subnormal subgroups are c-normal and the other in which all subnormal subgroups are c-permutable, and prove some related properties.

8. To establish two minimal non-X-groups: one where X is the property of c-normality being transitive and the other where X is the property of c-permutability being transitive.

1.5 Research Methodology

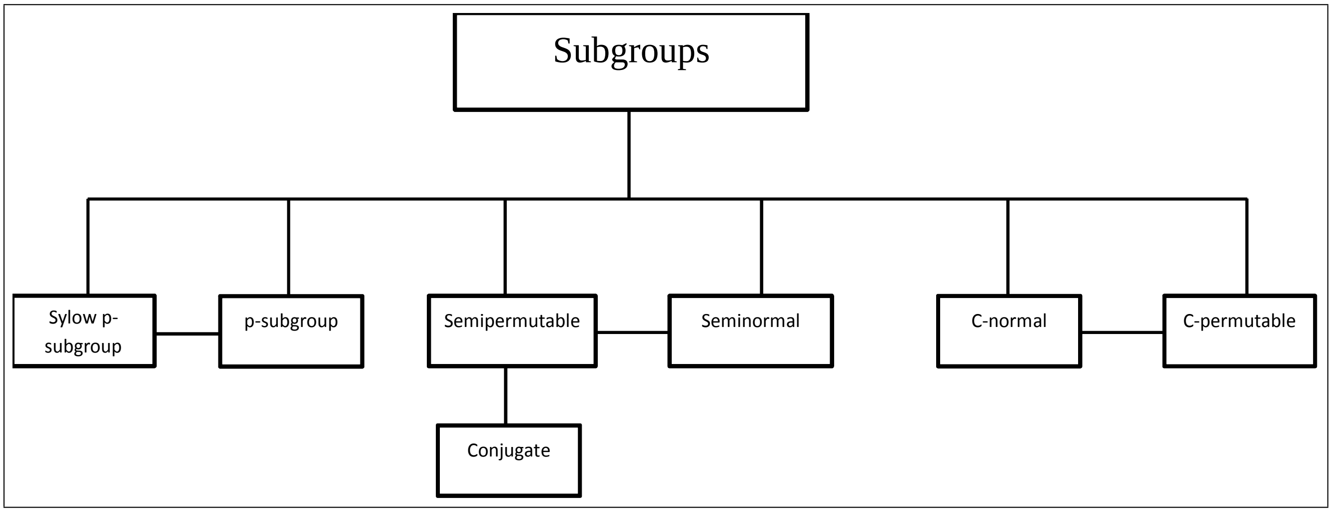
The research starts by studying some known subgroups of a finite group such as permutable, S-permutable, semipermutable, c-normal, c-permutable subgroups and their properties. These concepts have been a subject of interest to many researchers. This can been shown from results obtained from Al-Sharo et al. [4], Beidleman et al. **[11]**, Wang et al. [32], Beidleman et.al. [12], Agrawal [1], Ore [23] and some others. These papers are referred to in solving the main problems in this study. The basic definitions and the theorems in Chapter 2 are needed and will be used in much of the discussion in the following chapters. Further results on some known groups will be given and proven and some results will be used to prove other facts or theorems. In addition to that, structure of some new groups established via transitivity properties, minimal non-groups and other relation will be given with proof.

A summary of the research methodology is shown in Figure 1.1. The description of the work methodology is divided into two major parts: (a) groups on transitivity and (b) types of subgroups. "Proposed Group" indicates a new class of groups that will be introduced in this thesis.



(a)

Figure 1.1.a: Flow chart of research relation and work methodology between groups created from transitivity.



(b)

Figure 1.1.b: Flow chart of research relation and work methodology between subgroups

1.6 Organization of the Thesis

This thesis is organized in seven chapters. **Chapter 1** provides the introduction of this thesis. It also explains the general background and literature review of the study, the problem statement, the scope and objectives, the research methodology of this work and the organization of this thesis.

Chapter 2 contains preliminaries and important concepts in group theory relevant to the study. It also includes notations, terminologies and basic results that are needed for the whole thesis, as well as some important definitions. This chapter also contains descriptions on the T-groups and PT-groups and more detailed descriptions on the PST-groups, S-permutably embedded subgroups, CPT-groups, and other known groups and related concepts.

Chapter 3 contains new results on p-subgroups, S-permutable subgroups, and S-permutably embedded subgroups together with the proofs. New characterization on PST-groups in relation to the S-permutably embedded subgroups are also established with proofs. In addition, similar properties are shown for a new class of groups equivalent to PST-groups but with an additional condition on the nilpotent residual.

In **Chapter 4,** new results are proven on semipermutable subgroups. This chapter also includes a proof that the Dihedral group is always a BT-group. An example of an SP-group with the smallest order, that is not a BT-group is also shown in this chapter.

Chapter 5 introduces a new class of groups where c-normality is a transitive relation and provide examples to clarify these groups. Moreover, some theorems and lemmas about these groups in relation to supersolvable subgroups, direct product and nilpotent residual are stated and proven. This chapter also includes a counter example to show that c-normal subgroups and c-permutable subgroups are not equivalent.

In Chapter 6, new results on CPT-groups with X_p-groups and S-permutably embedded subgroups are established and proven. In addition to that, this chapter also includes new concepts of groups in which all subnormal subgroups are c-normal, groups in which all subnormal subgroups are c-permutable and non-X-groups for two properties of X; one is the property where c-normality is transitive and the other is the property where c-permutability is transitive. New theorems on these groups are also stated and proved.

Chapter 7 summarizes the results of this study. This is followed by a number of suggestions and open problems for future research.

1.7 Summary

This chapter briefly explained the background and literature review of the study. Among others, it also stated the scope as well as the objectives of the study. Lastly, it described the contents of each chapter.

Chapter 2

PRELIMINARIES

2.1 Introduction

In the previous chapter, a brief introduction on some known groups such as permutable, semipermutable, c-normal and other groups were given. This chapter presents some preliminaries needed throughout the thesis in addition to the theorems and definitions that are needed in the explanation of results. This chapter begins with a review of some preliminaries on group theory. Then it provides some elementaries on nilpotent, solvable, and supersolvable groups and other groups in Section 2.3. In Section 2.4, some definitions and facts about T-, PT-, and PST-groups are given. Section 2.5 introduces some facts about X_p-groups and C_p-condition. The normally embedded and S-permutably embedded subgroups are introduced in Section 2.6. In Section 2.7, semipermutable subgroups, BT-groups and SP-groups as well as some related theorems are reviewed. Following that is Section 2.8, which introduces the c-normal subgroups. The last section is about c-permutable subgroups and CPT-groups.

2.2 Preliminaries on Group Theory

In order to study an abstract group, it will be helpful to compare the group with a specific known group. This comparison is carried out by a function called homomorphism. In the following, the property of homomorphism is given.

Theorem 2.1 *[19] Let $\phi : G \to G'$ be a group homomorphism. Then the following holds:*

(i) If $H \leqslant G$ then $\phi(H) \leqslant G'$.

(ii) If $K' \leqslant G'$ then $\phi^{-1}(K') \leqslant G$.

(iii) If $H \trianglelefteq G$ then $\phi(H) \trianglelefteq G'$.

The following proposition is called the Zassenhaus - butterfly Lemma, which is one of the important facts in group theory, since many theorems has been built on it.

Proposition 2.2 *[19]* ***(Zassenhaus – butterfly Lemma)*** *Let H and K be subgroups of a group G and let H^* and K^* be normal subgroups of H and K respectively, Then:*

(i) $H^*(H \cap K^*)$ is a normal subgroup of $H^*(H \cap K)$.

(ii) $K^*(H^* \cap K)$ is a normal subgroup of $K^*(H \cap K)$.

(iii) $H^*(H \cap K)/H^*(H \cap K^*) \cong K^*(H \cap K)/K^*(H^* \cap K)$

$$\cong (H \cap K)/[(H^* \cap K)(H \cap K^*)].$$

The following definitions and theorems are some background knowledge needed in this study.

Definition 2.3 *[22] A composition series of a group G is a normal series*

$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_{r-1} = G$, *such that* G_{i+1}/G_i *is simple (and non-trivial) for* $1 \leqslant i \leqslant r$-1, *and the subgroup* G_{i+1}/G_i *is called the composition factors.*

Theorem 2.4 *[19] **(Jordan Holder -Theorem)** Any two composition series of a given group are equivalent. They have the same composition factors.*

Definition 2.5 *[19] A group G is said to be a p-group if every element in G has order a power of the prime p. A subgroup of a group G is called a p-subgroup in G if the subgroup itself is a p-group.*

Remark 2.6 *Let G be a finite group. Then G is a p-group if and only if $|G| = p^n$ for some prime p.*

Definition 2.7 *[19] A Sylow p-subgroup of a group G is a maximal p-subgroup of G. It is denoted the set of all Sylow p-subgroups of G by $Sly_p(G)$.*

Definition 2.8 *[19] Let H be a non-empty subset of group G and let $g \in G$. The conjugate of H in G is the set $H^g = \{ghg^{-1} : h \in H\}$.*

Definition 2.9 *[21]Let H be a non-empty subset of group G, the normalizer of H in G is the set $N_G(H) = \{x \in G : x^{-1}Hx = H\}$.*

The normalizer of H is a subgroup containing H, and it is the largest subgroup of G in which H is normal.

Definition 2.10 *[28] A minimal normal subgroup N of G is a normal subgroup $\neq 1$ that contains no proper subgroup that is normal in G, and denoted by $N \triangleleft G$. i.e The only normal subgroup contained in N are N and 1.*

Definition 2.11 *[21] If there is a finite chain of subgroups of the group G, each one normal in the next, beginning at H and ending at G, then H is called subnormal in G and denoted by $H \lhd\lhd G$.*

Not every subnormal subgroup is normal as in the following example.

Example 2.12 *Let $G = D_8 =< r, s|r^4 = s^2 = 1, rs = sr^{-1} >$*

Note that $< s > \lhd\lhd D_8$ but $< s > \ntriangleleft D_8$.

The following proposition states some basic properties of subnormal subgroups.

Proposition 2.13 *[21], [27]*

(i) If $A \lhd\lhd G$ and $B \lhd\lhd G$ then $A \cap B \lhd\lhd G$ and $\langle A, B\rangle \lhd\lhd G$.

(ii) Let $\phi : G \to G'$ be homomorphism if $H \lhd\lhd G$ then $\phi(H) \lhd\lhd \phi(G)$.

(iii) If $H \lhd\lhd G$ and $K \leqslant G$, then $H \cap K \lhd\lhd K$.

(iv) If $H \lhd\lhd G$, G finite group and M is minimal normal subgroup of G then $M \subset N_G(H)$.

(v) Let $K \leqslant H \leqslant G$, if $K \lhd\lhd H$, and $H \lhd\lhd G$, then $K \lhd\lhd G$.

(vi) Let G a finite p-group, then every subgroup of G is subnormal in G.

(vii) If $H \lhd\lhd G$ and $K \lhd G$ then $HK \lhd\lhd G$.

The following proposition will be used in Example 3.5.

Proposition 2.14 *[19] If $G = A \times B$ then $\overline{A} = A \times \{e\} \approx A \trianglelefteq G$.*

To determine the structure of a finite group, some special groups and subgroups are usually considered, such as the commutators, the automorphisms, the characteristic groups and the Sylow p-subgroups. Some facts and definitions about these subgroups are given below.

Definition 2.15 *[22] The commutator of the ordered pair of elements a, b in the group G is the element $a^{-1}b^{-1}ab$. It is denoted by $[a, b]$.*

Definition 2.16 *[22] The derived group (or commutator subgroup) of G is the subgroup generated by all commutator elements in G . It is denoted by G' .*

$$i.e\ G' = [G, G] = \langle a^{-1}b^{-1}ab | a, b \in G \rangle.$$

Definition 2.17 *[17] An isomorphism from G to G is called an automorphism. The set of all such automorphisms is denoted by $Aut(G)$.*

Remark 2.18 *$Aut(G)$ is a subgroup of S_G.*

Definition 2.19 *[17] A subgroup H of a group G is called a characteristic of G denoted by $H\ Char\ G$ if every automorphism of G maps H to it self. i.e $\sigma(H) = H, \forall \phi \in Aut(G)$.*

Some examples on characteristic group will be illustrated in the next paragraphs. Following that is a proposition containing equivalent statements regarding the Sylow p-subgroups in relation to the characteristic group.

Example 2.20 *(i) $Z(G)\ Char\ G$.*

(ii) $\phi(G) = \cap\{M : M \text{ is maximal subgroup in } G\}$, then $\phi(G)$ is characteristic in G.

Proposition 2.21 *[27]Let $P \in Syl_p(G)$. Then the following are equivalent:*

(i) $P \trianglelefteq G$

(ii) P is a unique Sylow p-subgroup of G.

(iii) $P\ Char\ G$.

The p-core, and $p^{'}$-core of finite groups are defined as:

Definition 2.22 *[31] For a prime p, the p-core of a finite group is defined to be its largest normal p-subgroup. It is the normal core of every Sylow p-subgroup of the group. The p-core of G is often denoted $O_p(G)$.*

Definition 2.23 *[31] For a prime p, the p'-core is the largest normal subgroup of G whose order is coprime to p and is denoted $O_{p'}(G)$.*

The following proposition is the Dedekind law and Dedekind definition.

Proposition 2.24 *[21] Let H and K be subgroups of a group G and let $H \subseteq U \subseteq G$ where U is also a subgroup. Then $HK \cap U = H(K \cap U)$.*

Definition 2.25 *[22]Dedekind group is a group G such that every subgroup of G is normal.*

Definition 2.26 *[22] A simple group is a group in which no proper subgroup is normal.*

Abelian groups are examples of Dedekind groups. On the other hand, the quaternion group Q_8 is a non-abelian Dedekind group.

In group theory, the concept of a semidirect product is a generalization of a direct product. Furthermore, there are two closely related concepts of semidirect product: an inner semidirect product is a particular way in which a group can be constructed from two subgroups, one of which is a normal subgroup, while an outer semidirect product is a cartesian product as a set, but with a particular multiplication operation. Next is the definition of semidirect product.

Theorem 2.27 *[17] Let H and K be groups and let φ be a homomorphism from K into $Aut(H)$. Then $\bullet$ denote the left action of K on H determined by φ. Let G be the set of ordered pair $(h,k)$with $h \in H$ and $k \in K$ and define the following multiplication on G:*

$$(h_1,k_1)(h_2,k_2) = (h_1 k_1 \bullet h_2, k_1 k_2)$$

(i) This multiplication makes G into a group of order $|G| = |H||K|$

(ii) The set $\{(h,1)|\ h \in H\}$ and $\{(1,k)|\ k \in K\}$ are subgroups of G and the map $h \to (h,1)$ for $h \in H$ and $k \to (1,k)$ for $k \in K$ are isomorphisms of these subgroups with the groups H and K respectively: $H \cong \{(h,1)|\ h \in H\}$ and $K \cong \{(1,k)|\ k \in K\}$.

Identifying H and K with their isomorphic copies in G described in (ii) we have

(iii) $H \trianglelefteq G$.

(iv) $H \cap K = 1$.

(v) for all $h \in H$ and $k \in K, khk^{-1} = k \bullet h = \varphi(k)(h)$

Definition 2.28 *[17] Let H and K be groups and let φ be a homomorphism from K into $Aut(H)$. The group described in Theorem 2.27 is called a semidirect product of H and K with respect to φ and it is denoted by $H \rtimes K$.*

Proposition 2.29 *[17] Let H and K be a groups and let $\varphi : K \to Aut(H)$ be a homomorphism. Then the following are equivalent:*

(i) The identity (set) map between $H \rtimes K$ and $H \times K$ is a group homomorphism.

(ii) φ is the trivial homomorphism from K into $Aut(H)$.

(iii) $K \trianglelefteq H \rtimes K$.

Next is an example on semidirect product.

Example 2.30 *The symmetric group of order 6, S_3, is isomorphic to the semidirect product $Z_3 \rtimes Z_2$.*

Definition 2.31 *[25] An automorphism of a group G that leaves every subgroup invariant is called a power automorphism.*

In the following section, some definition and theorems on nilpotent and solvable groups are given, which will be used in the study.

2.3 Nilpotent and Solvable Groups

Nilpotent groups might be considered as a generalization of p-groups, also many of the properties of finite p-groups are shared with the class of nilpotent groups. This section lists some properties of nilpotent residual groups, Hall groups, supersolvable and other classes of groups. These properties will be needed in the sections that follows.

Definition 2.32 *Let G be a group, and $\mathfrak{x}$ be a class of group. Then $\mathfrak{x}$-residual of G is defined by:*

$$G = \cap\{N \trianglelefteq G : G/N \in \mathfrak{x}\}$$

Example 2.33 *(i) Let $x = a$, be the class of abelian groups, and let $G = S_3$, Then the normal subgroups of S_3 are :$\{1, A_3, S_3\}$, so $S_3/1 \approx S_3 \notin a$, $S_3/A_3 \approx Z_2 \in a$, and $S_3/S_3 \approx 1 \in a$, with $\cap\{A_3, S_3\} = A_3$, so $S_3^a = A_3$.*

(ii) Let $x = \eta$, be the class of nilpotent groups, and let $G = S_3$, Then the normal subgroups of S_3 are : $\{1, A_3, S_3\}$,so $S_3/1 \approx S_3 \notin \eta$, $S_3/A_3 \approx Z_2 \in \eta$,and $S_3/S_3 \approx 1 \in \eta$, with $\cap\{A_3, S_3\} = A_3$, so $S_3^\eta = A_3$.

(iii)Let $\mathfrak{x} = \eta$, be the class of nilpotent groups, and let $G = S_4$, Then the normal subgroups of S_4 are: $\{1, V_4, A_4, S_4\}$,so $S_4/1 \approx S_4 \notin \mathfrak{n}$, $S_4/V_4 \approx Z_6 \in \eta$,$S_4/A_4 \approx Z_2 \in \eta$, and $S_4/S_4 \approx 1 \in \eta$, with $\cap\{V_4, A_4, S_4\} = V_4$, so $S_4^\eta = V_4$.

Remark 2.34 *Nilpotent residual G^η is the smallest normal subgroup of G such that G/N is nilpotent.*

The definition of normal p-complement and p-nilpotent will be given next. Both definitions will be used in Chapter 6.

Definition 2.35 *[35]A normal p-complement of a finite group for a prime p is a normal subgroup of order coprime to p and index a power of p. In other words the group is a semidirect product of the normal p-complement and any Sylow p-subgroup.*

Definition 2.36 *[35]A group is called p-nilpotent if it has a normal complement.*

Definition 2.37 *[22] A subgroup whose order is relatively prime (coprime) to its index is called a Hall subgroup, i.e H is a Hall subgroup of G if $((G : H), |H|) = 1$.*

Definition 2.38 *[22] Let π be a set of primes, and π-number is an integer whose prime divisor all belong to π . The complement of π in the set of all primes is denoted by π'.*

The concept of π-number is used to define the Hall π-subgroups. Some preliminaries about Hall subgroups are introduced next.

Definition 2.39 *[22] Let π be the set of primes. A subgroup H of group G is called Hall π-subgroup if the order of H is a π-number and $(G : H)$ is a π'-number .*

Remark 2.40 *If $\pi = \{p\}$ then the Hall π-subgroups of a group G are the Sylow p-subgroups. To see this, let $|G| = p^n m$ with $p \nmid m$. If $P \in Syl_p(G)$ then $|P| = P^n$ which is $\pi = \{p\}$-number and $(G : P) = \frac{|G|}{|P|} = \frac{P^n m}{P^n} = m$, m is π'-number.*

The following example is for the previous remark.

Example 2.41 *Let $G = S_4$, and let H be a subgroup of G with isomorphic to V_4 i.e $H \approx V_4 \leq S_4$*

Now $|H| = 4$, and $(S_4 : H) = \frac{|S_4|}{|H|} = 6$ with $gcd(4, 6) = 2 \neq 1$. So H is not a Hall subgroup of S_4.

But $|S_4| = 2^3.3$ and $P \in Syl_2(S_4)$ implies $|P| = 2^3$.

$(S_4 : P) = \frac{2^3.3}{2^3} = 3$ and $P \in Syl_p(G)$ so P is a Hall subgroup.

The class of solvable groups is a class which is larger than the nilpotent class and it has many interesting properties. The definition and some basic properties of solvable groups are listed below.

Definition 2.42 *[25] A group G is said to be solvable (or soluble) if it has a series $1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_n = G$ in which each factor G_{i+1}/G_i is abelian.*

Note that every abelian group is solvable.

Theorem 2.43 *[17] Any subgroup H of a solvable group G is solvable.*

Theorem 2.44 *[17] (Burnside Theorem) If $|G| = p^a q^b$ for some primes p and q, then G is solvable.*

The definition and basic properties of the class of supersolvable groups are listed below.

Definition 2.45 *[25] A group G is said to be supersolvable (or supersoluble) if it has a normal cyclic series, i.e. a series of normal subgroups whose factors are cyclic.*

Note that every supersolvable group is solvable.

Theorem 2.46 *[14] Supersolvable groups are closed under passage to subgroups, quotients, and direct product.*

Theorem 2.47 *[5] Assume that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of H is normal in G . Then G is supersolvable.*

Theorem 2.48 *[20]Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersolvable.*

Theorem 2.49 *[14] Let G be a group with normal subgroups H and K. If G/H and G/K are supersolvable then $G/(H \cap K)$ is supersolvable.*

Theorem 2.50 *[14] Let G be any group and $N \trianglelefteq G$. If N is cyclic and G/N is supersolvable then G is supersolvable.*

Theorem 2.51 *[20] Every finite p-group is supersolvable.*

Theorem 2.52 *[20] Every finite nilpotent group is supersolvable.*

Remark 2.53 *For finite groups the following inclusion applies: Nilpotent $\subset$ Supersolvable $\subset$ Solvable.*

According to Ballester-Bolinches[**10**], Doerk determined the structure of minimal non-supersolvable groups (a non-supersolvable group has the property that all of its proper subgroups are supersolvable).

The following theorem is regarding the minimal non-supersolvable group.

Theorem 2.54 *[10] Let G be a minimal non-supersolvable group, then G has a unique normal Sylow subgroup P.*

Next are definitions of Frattini group and Fitting group.

Definition 2.55 *[20] The intersection of all maximal subgroups of a group G is called the Frattini subgroup of G denoted by $\Phi(G), i.e \Phi(G) = \bigcap_{\max M} M$.*

Theorem 2.56 *[20] $\Phi(G)$ is nilpotent for any group G.*

Definition 2.57 *[20]The Fitting subgroup of a group is defined as the unique largest normal nilpotent subgroup of G denoted by $Fit(G)$.*

The next two theorems are on Fitting and Frattini subgroups of a group G and following these some relations between these two concepts are given.