SOLUTION OF CONVECTIVE BOUNDARY LAYER FLOWS VIA SCALING GROUP TRANSFORMATION METHOD

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by

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DEDICATION

To my teachers, my parents, my brothers, my sisters and my wife, my gorgeous daughter Tibah and my gorgeous son Hussein.

Thank you for your unwavering love, continued encouragement and support. I will cherish you all the days of my life.

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(**In the name of Allah, The Most Gracious, Most Merciful**)

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Nomenclature

a	Momentum slip factor
A	Thermal conductivity parameter
b	Dimensionless thermal slip parameter
B	Convective heat parameter
c_i	Constant
c_p	Specific heat at constant pressure of the fluid
C_f	Friction factor
D_1	Thermal slip factor with dimension length
Ec	Dissipation parameter (Eckert Number)
f	Dimensionless stream function
f_w	Suction /injection parameter
g	Acceleration due to gravity
Gr	Grashof number
h_f	Heat transfer coefficient
I	Vortex viscosity parameter
j	Microinertia density
K	Permeability for the porous medium
K_1	Consistency coefficient of the fluid
k_1	Mean absorption coefficient
k	Thermal conductivity
k_p	Constant permeability of the porous medium
k_∞	Constants (ambient temperature)
L	Characteointic length
M	Viscosity variation parameter
m	Falkner-Skan flow parameter
n	Power-law index
N	Angular velocity

$Nu_{\bar{x}}$	Local Nusselt number
Pr	Prandtl number
Q	Heat generation parameter
Q_0	Heat generation constant
q_r	Radiative heat flux
R	Radiation parameter
Re	Reynolds number based on characteristic length
Re_x	Local Reynolds number
T	Fluid temperature
T_w	Temperature of the fluid at the wall
T_∞	Temperature of the fluid at infinity
T_f	Hot fluid temperature
U_r	Reference velocity
U_∞	Free stream velocity
$\bar{u}$	Velocity along the plate
u_w	Velocity of the plate
$\bar{u}_w(\bar{x})$	Wedge velocity
$\bar{u}_e(\bar{x})$	Free stream velocity
$\bar{v}$	Velocity normal to the plate
$\bar{v}_w(\bar{x})$	Suction/Injection velocity
$\bar{x}$	Coordinate along the surface
$\bar{y}$	Coordinate normal to the surface

Greek symbols

α	Thermal diffusivity
τ	Shear stress
τ_w	Shearing stress on the surface
β	Coefficient of thermal expansion
β_1	Hartee pressure gradient parameter
δ	Velocity slip parameter
η	Similarity independent variable

θ	Dimensionless temperature
ρ	Density of the fluid
μ	Dynamic viscosity
$\bar{\mu}$	Viscosity of the fluid
$\mu_\circ$	Ambient fluid dynamic viscosity
Ω	Rheological parameter
υ	Kinematic viscosity
ε	Power for of scaling group of transformations.
σ_1	Stefan-Boltzmann constan
ψ	Stream function
γ	Micropolar spin gradient viscosity
λ	Micro-rotational density parameter
Δ	Micro-polar parameter
κ	Micropolar vortex viscosity
ω	Wedge velocity parameter

BEBERAPA PENYELESAIAN BAGI ALIRAN LAPISAN SEMPADAN OLAKAN MELALUI KAEDAH PENJELMAAN KUMPULAN PENSKALAAN

ABSTRAK

Tujuan tesis ini adalah untuk mencari penyelesaian keserupaan bagi beberapa model dalam mekanik bendalir menggunakan kaedah penjelmaan kumpulan penskalaan. Dalam tesis ini aliran lapisan sempadan berlamina dua dimensi tak boleh mampat yang mantap pada plat rata dan baji di dalam media berliang dan tidak berliang telah dikaji. Bendalir yang dipertimbangkan ialah Newtonan dan bukan Newtonan (bendalir hukum kuasa, mikrokutub) dengan syarat sempadan gelinciran, radiasi haba, kelikatan boleh ubah, syarat sempadan olakan dan kesan penjanaan haba. Kaedah penjelmaan kumpulan penskalaan dapat mengurangkan bilangan pemboleh ubah tak bersandar dan pemboleh ubah bersandar serta memetakan persamaan pembezaan separa kepada persamaan pembezaan biasa. Dengan menggunakan analisis penjelmaan simetri, penjelmaan keserupaan diperoleh. Oleh itu, persamaan menakluk bagi model aliran dijelmakan menjadi persamaan keserupaan. Persamaan tersebut diselesaikan secara berangka menggunakan kaedah Runge-Kutta-Fehlberg keempat-kelima dengan Maple 13. Kesan parameter (gelinciran halaju, gelinciran terma, hukum kuasa Falkner-Skan, keberaliran haba, kelikatan boleh ubah, sedutan/suntikan, indeks hukum kuasa, radiasi haba, indeks kelikatan, reologi, penjanaan haba, perolakan haba, nombor Prandtl, mikrokutub, nombor Grashof, kebolehtelapan, kepadatan putaran mikro, nombor Eckert, kelikatan vorteks dan putaran mikro) pada halaju, halaju sudut, suhu dan kuantiti fizikal (tegasan ricih, kadar pemindahan haba dan faktor regangan pasangan) yang tak berdimensi telah dikaji dan dibincangkan. Keputusan berangka daripada kajian ini menepati keputusan yang telah diterbitkan sebelum ini **(yang tersedia)**. Kajian ini boleh diaplikasikan dalam injap jantung tiruan, rongga dalaman, peranti nano/mikro, serat sintetik, bahan makanan, penyemperitan plastik lebur dan dalam beberapa aliran larutan polimer.

SOLUTIONS OF CONVECTIVE BOUNDARY LAYER FLOWS VIA SCALING GROUP TRANSFORMATION METHOD

ABSTRACT

The aim of this thesis is to find similarity solutions for some models in fluid mechanics using the method of scaling group transformation. In this thesis the steady two-dimensional incompressible laminar boundary layer on a flat plate as well as wedge flow models both in porous media and clear media have been studied. The working fluid considered is Newtonian and non-Newtonian (power-law, micropolar) with slip boundary condition, thermal radiation, variable viscosity, convective boundary condition and heat generation effects. Scaling group transformation method reduces the number of the independent variables as well as the dependent variables and maps the partial differential equations to ordinary differential equations. Using the symmetry transformations analysis, the similarity transformations have been obtained. Hence, the governing equations for flow models are transformed into similarity equations. The transformed equations are solved numerically by the Runge-Kutta-Fehlberg fourth-fifth order numerical method using Maple 13. The effects of parameters (velocity slip, thermal slip, Falkner-Skan power-law, thermal conductivity, variable viscosity, suction/injection, power-law index, thermal radiation, viscosity index, rheological, heat generation, convective heat, Prandtl number, micropolar, Grashof number, permeability, micro-rotational density, Eckert number, vortex viscosity and microrotation) on the dimensionless velocity, angular velocity, temperature and the physical quantities (shear stress, heat transfer rate and couple stress factor) have been studied and discussed. Good agreements were found between the numerical results of the present study with published results (where available). The present study find applications in artificial heart valves, internal cavities, nano/micro devices, synthetic fibers, foodstuffs, extrusion of molten plastic and as well as in some flows of polymer solutions.

CHAPTER 1

GENERAL INTRODUCTION AND BASIC CONCEPTS

This chapter gives a general background on fluid flow, boundary layers, heat transfer and solution methods. It will include basic concepts and definitions used in this study. It will also include types of fluids, governing equations, commonly used dimensionless numbers, the use of group transformation and the numerical method that will be used in this study. This chapter will also discuss the objectives, methodology and scope of the research that has been conducted.

1.1 Types of Fluids

1.1.1 Newtonian and Non-Newtonian Fluids

Fluids such as water and air are described as Newtonian fluids. These fluids are essentially modelled by the Navier-Stokes equations which describe a linear relation between the stress and the strain rate.

On the other hand, there are a large number of fluids that do not fall in the category of Newtonian fluids and are called non-Newtonian fluids. Examples include toothpaste, egg whites and liquid soaps. A distinguishing feature of many non-Newtonian fluids is that they exhibit both viscous and elastic properties and the relationship between the stress and the strain rate is non-linear. Contrary to Newtonian fluids, there is not a single model that can describe the behavior of all the non-Newtonian fluids and many models have been proposed such as, micropolar, viscoelastic, power-law, Carreau, Eyring, Ellis and Herschel–Bulkley fluid models.

1.1.2 Micropolar Fluids

Micropolar fluids are fluids with microstructure belonging to a class of fluids with nonsymmetrical stress tensor referred to as polar fluids. Physically, they represent fluids consisting of randomly oriented particles suspended in a viscous medium (Lukaszewicz, 1999). The theory of micropolar fluid (which is a special form of non-Newtonian fluid) includes the effect of micro-inertia and couple stresses. The theory explains the non-Newtonian behavior of certain polymeric fluid, animal blood and liquid crystals (Parmar and Timol, 2012).

1.1.3 Power-law Fluid Model

A power-law fluid, or the Ostwald-de Waele relationship, is a type of generalized Newtonian fluid for which the shear stress τ is given by (Bird et al., 1987):

$$\tau = K\left(\frac{\partial u}{\partial y}\right)^{n}, \qquad (1.1)$$

where K is the flow consistency index, $\frac{\partial u}{\partial y}$ is the shear rate or the velocity gradient perpendicular to the plane of shear and n is the flow behavior index (dimensionless). The quantity $\mu_{eff} = K\left(\frac{\partial u}{\partial y}\right)^{n-1}$, represents the apparent or effective viscosity as a function of the shear rate. Power-law fluids can be subdivided into three different types of fluids based on the value of their flow behavior index: when $n<1$ the fluid is called pseudoplastic or shear-thinning fluid, when $n=1$ the fluid is called Newtonian fluid and when $n>1$ the fluid is called dilatant or shear-thickening fluid. The relationship between shear rate and shear stress is illustrated in Figure 1.1.

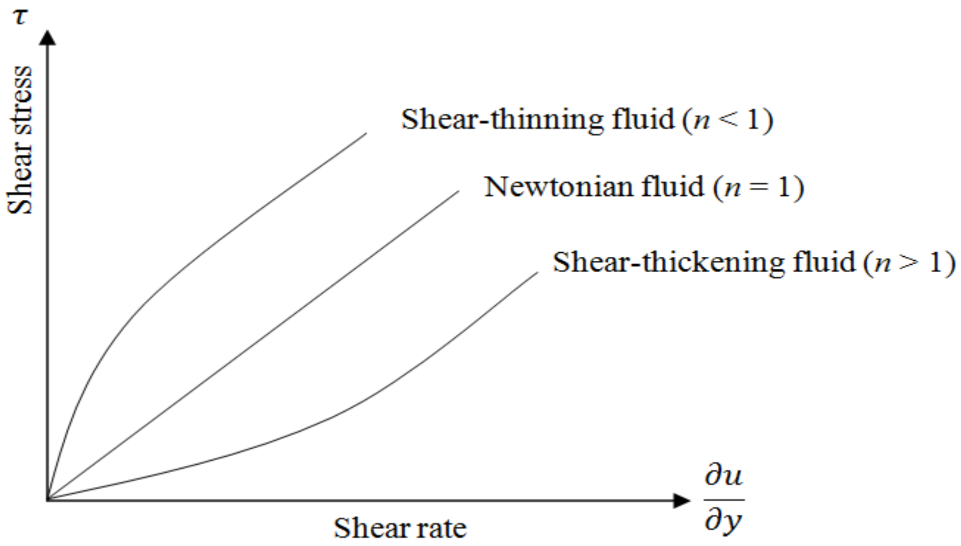

Figure 1.1: Flow curves of power-law fluids (Uddin, 2013).

1.2 Fluid Flows

1.2.1 Unsteady and Steady Flows

A flow whose flow state expressed by all fluid flow properties (e.g., velocity, temperature, pressure, and density) at any position, does not change with time, is called a steady flow. On the other hand, a flow whose flow state does change with time is called an unsteady flow (Bansal, 2005).

1.2.2 Laminar and Turbulent Flows

Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth streamlines is called laminar. The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities characterized by velocity fluctuations is called turbulent. The flow of low-viscosity fluids such as air at high velocities is typically turbulent. The flow regime greatly influences the heat transfer rates and the required power for pumping (Cengel, 2006).

1.2.3 Compressible and Incompressible Flows

An incompressible flow is the type of flow in which the variation of the mass per unit volume (density) within the flow is constant. In general, all liquids are treated as the incompressible fluids. On the contrary, flows which are characterized by a varying density are said to be compressible. Gases are normally compressible fluids (Bansal, 2005).

1.3 Heat Transfer

Heat transfer is the branch of engineering science which seeks to predict the energy transfer which may take place between material bodies as a result of temperature difference. Due to temperature difference, heat flows from the region of high temperature to the region of low temperature **(**Borthakur and Hazarika, 2010).

Heat transfer is applied in various aspects of engineering. Electrical engineers apply their knowledge of heat transfer to design cooling system for motors, generators and transformers. The mechanical engineer deals with the problem of heat transfer in the field of internal combustion engines, steam generation, refrigeration and heating and ventilation. In the design of heat exchangers such as boilers, condensers, radiators etc., heat transfer analysis is essential for sizing such equipments. In heating and air conditioning applications for buildings, a proper heat transfer analysis is necessary to estimate the amount of insulation needed to prevent excessive heat losses or gains. Chemical engineers are concerned with the evaporation, condensation, heating and cooling of fluids. In the design of nuclear reactor a thorough heat transfer analysis of fuel elements is important for proper sizing of fuel elements to prevent burnout. In aerospace technology, the temperature distribution and heat transfer are crucial because of weight limitation and safety consideration (Borthakur and Hazarika, 2010).

There are three different modes of heat transfer: conduction, convection and radiation. In reality, temperature distribution in a medium is controlled by the combined effects of these three modes of heat transfer; therefore, it is not possible to isolate entirely one mode from interacting with the other modes (Borthakur and Hazarika, 2010).

1.3.1 Convective Heat Transfer

Convection or convective heat transfer is one of the modes of heat transfer besides conduction and radiation. This mode of heat transfer is met with in situations where energy is transferred as heat to a flowing fluid at the surface over which the flow occurs. This mode is basically conduction in a very thin fluid layer at the surface and then mixing caused by the flow. The energy transfer is by combined molecular diffusion and bulk flow (Kaothandaraman, 2010). The heat flow is independent of the properties of the material of the surface and depends only on the fluid properties. However, the shape and nature of the surface will influence the flow and hence the heat transfer. Convection is not a pure mode as conduction or radiation and hence involves several parameters. If the flow is caused by external means like a fan or pump, then the mode is known as forced convection. If the flow is due to the buoyant forces caused by temperature difference in the fluid body, then the mode is known as free or natural convection. In most applications, heat is transferred from one fluid to another separated by a solid surface. So heat is transferred from the hot fluid to the surface and then from the surface to the cold fluid by convection. In the design process thus convection mode becomes the most important one from the point of view of application (Kaothandaraman, 2010).

1.4 Boundary Layer Concept

Boundary layer theory is the cornerstone of our knowledge of the flow of air and other small viscosity under circumstance of interest in many engineering applications. Thus, many complex problems in aerodynamics have been clarified by a study of the boundary layer and its effect on the general flow around the body.

One of the most convenient concepts in fluid mechanics is that which classifies the flow about solid bodies into two regions. The first is the main stream in which the ideal frictionless fluid theory can be successfully employed. The second is the boundary layer adjacent to the solid surface in which viscous effects are equally important to inertia effects. With the aid of the idealization due to Prandtl number will be defined later, many flow fields may be mathematically modelled and deductions made which correspond well with experimental by observed results.

Boundary layer theory was developed mainly for the case of laminar flow in an incompressible fluid. Later, the theory was extended to include turbulent and incompressible boundary layers, which are more important from the point of view of practical applications (Schlichting and Gersten, 2000).

Ludwig Prandtl in 1904 introduced the concept of a boundary layer in large Reynolds number flows and he also showed how the Navier-Stokes equations could be simplified to yield approximate solutions. There are many books on boundary layer theory for example, by Schlichting (1979), Schlichting and Gersten (2000) and Naz (2008).

A boundary layer is a thin layer in which the effect of viscosity is important no matter how high the Reynolds number may be. The Reynolds number Re will be defined later. A boundary layer exists if $\sqrt{Re} >> 1$.

A boundary layer does not necessarily need to be adjacent to a solid boundary. A thin region of sharp change can exist away from a boundary such as along the axis of a free jet. The boundary layer equations are applicable in the thin region of sharp change (Naz, 2008).

Boundary layers of non-Newtonian fluids have received considerable attention in last decades. Boundary layer theory has been applied successfully to various non-Newtonian fluids models. One of these models is the power-law fluid; the first considered the form of the boundary-layer equations for a power-law fluid by Schowalter (1960) and Acrivos et al. (1960). Schowalter (1960) derived the equations governing the self-similar flow of a pseudo-plastic fluid and Acrivos et al. (1960) provided numerical solutions to the equations governing the self-similar flow for both shear-thinning and shear-thickening fluids (Denier and Dabrowski, 2004).

The theory of micropolar fluids was first introduced by Eringen (1964, 1966, 1972). In this theory, the micropolar fluid exhibits the microrotational effects and micro-inertia. It is applied to describe the non-Newtonian behavior of certain fluids, such as liquid crystals, ferro liquids, colloidal fluids, and liquids with polymer additives. Many attempts were made to find analytical and numerical solutions, applying certain special conditions and using different mathematical approaches. The study of micropolar fluid mechanics has received the attention of several researchers. The boundary layer concept in micropolar fluids was studied by Willson (1969). A review of this study was provided by Ariman et al. (1973, 1974). A similarity solution was provided for the micropolar boundary layer flow over a semi-infinite flat plate by Ahmadi (1976). Hassanien et al. (1999) have studied flow and heat transfer in boundary layer of a micropolar fluid on a continuous moving surface. An excellent account of the theory and applications of boundary layer modeling has been given by Schlichting and Gerstern (2000).

1.4.1 Velocity Boundary Layer

The velocity boundary layer develops when there is fluid flow over a surface. Consider the flow with velocity u_∞ over a flat plate as shown in Figure 1.2. Beginning at the leading edge of the plate, the thickness of the boundary layer δ_v increases with the distance x, i.e. when measured along the surface. The region between the surface and the dash curve is the boundary layer or the velocity boundary layer (also called the hydrodynamic boundary layer) where the effects of viscosity are observed. Outside the boundary layer, i.e. the free stream flow, the viscosity is neglected. The velocity boundary layer ends at some arbitrary value of y where the velocity attained 99% of the free-stream velocity (Welty et al., 2008). Incropera and Dewitt (1985) have clearly described the mechanism of the flow to form the velocity boundary layer as follow: When the fluid particles make contact with the surface, they attain zero velocity. These particles then act to retard the motion of particles in the adjoining fluid layer, which act to retard the motion of particles in the next layer, and so on until, at a distance $y = \delta_v$ from the surface at this point the effect becomes negligible. This retardation of fluid motion is associated with shear stresses τ acting in planes that are parallel to the fluid velocity. With the increasing distance y from the surface, the x velocity component of the fluid u must increase until it approaches the free stream value u_∞. The subscript ∞ is used to designate conditions in the free stream outside the boundary layer. The boundary layer velocity profile refers to the manner in which u varies with y through the boundary layer (Bergman et al., 2011).

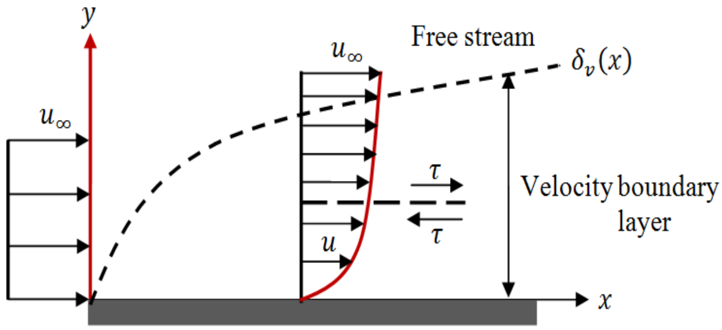

Figure 1.2: Velocity boundary layer.

1.4.2 Thermal Boundary Layer

The thermal boundary layer forms when there is temperature difference between the fluid stream and the surface (Incropera and Dewitt, 1985). Consider the fluid flow with velocity u_∞ and temperature T_∞ over a flat plate in Figure 1.3. The region from the plate surface to the curve is the thermal boundary layer. From the leading edge we can see that the thermal boundary layer thickness δ_T also increase along the flow direction x, following the trend of velocity boundary layer. This is mainly because of the effects of heat transfer into the free stream. The thermal boundary layer thickness can be defined as the distance from the surface in the y direction where $(T_w - T) = 0.99(T_w - T_\infty)$. It should be noted that T is the fluid temperature within the thermal boundary layer, T_w is the surface temperature and T_∞ is the free stream fluid temperature (Bergman et al. 2011).

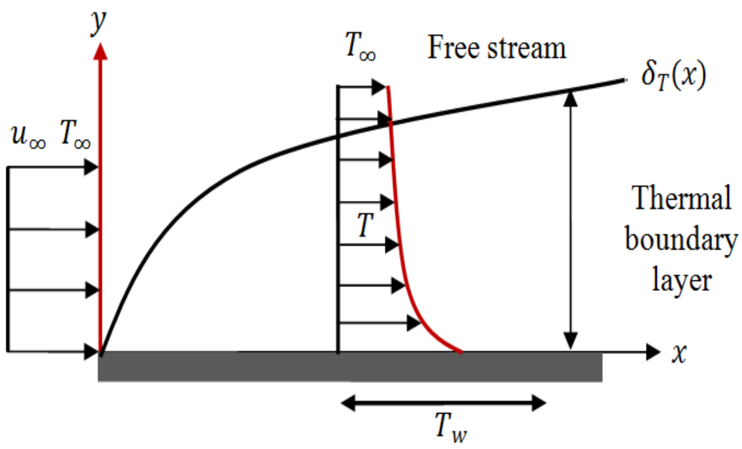

Figure 1.3: Thermal boundary layer.

Following the no-slip condition when the flow comes in contact with the surface of the plate, the temperature of the fluid will be equal to the surface temperature. Then, that layer of flow will change energy with the particles in the adjoining fluid layer which in turn change the energy with subsequent layer and from the process the temperature profile will develop (Rudramoorthy and Mayilsamy, 2006).

1.4.3 Basic Boundary Layer Equations

In this section, we show the basic equations of fluid mechanics, continuity, momentum, and energy and angular momentum equations for laminar, incompressible, two-dimensional flow of Newtonian, non-Newtonian power-law and micropolar fluid models.

Newtonian fluid model (White, 1998).

Continuity:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1.2}$$

Momentum:

$$\left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{x}} + \nu \left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right), \tag{1.3a}$$

$$\left(\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{y}} + \nu \left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right), \tag{1.3b}$$

Energy:

$$\rho c_p \left(\bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} \right) = k \left(\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right). \tag{1.4}$$

Non-Newtonian power-law fluid model (Dabrowski, 2009)

Continuity**:**

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1.5}$$

Momentum:

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mu}{\rho} \left| \frac{\partial \bar{u}}{\partial \bar{y}} \right|^{n-1} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{1.6a}$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}}, \tag{1.6b}$$

Energy:

$$\rho c_p \left(\bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} \right) = k \left(\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right). \tag{1.7}$$

Micropolar fluid model (Borthakur and Hazarika, 2010)

Continuity:

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \tag{1.8}$$

Momentum:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right)=(\kappa+\mu)\frac{\partial}{\partial \bar{y}}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)+\kappa\frac{\partial \bar{N}}{\partial \bar{y}}, \tag{1.9}$$

Angular Momentum

$$\rho j\left(\bar{u}\frac{\partial \bar{N}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{N}}{\partial \bar{y}}\right)=-\kappa\left(2\bar{N}+\frac{\partial \bar{u}}{\partial \bar{y}}\right)+\gamma\frac{\partial^2 \bar{N}}{\partial \bar{y}^2}, \tag{1.10}$$

Energy:

$$\rho c_p\left(\bar{u}\frac{\partial T}{\partial \bar{x}}+\bar{v}\frac{\partial T}{\partial \bar{y}}\right)=k\left(\frac{\partial^2 T}{\partial \bar{x}^2}+\frac{\partial^2 T}{\partial \bar{y}^2}\right)+(\mu+\kappa)\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2, \tag{1.11}$$

where $\bar{u}$ and $\bar{v}$ are the velocity components along the $\bar{x}$ and $\bar{y}$ axes, ν is the kinematic viscosity of the fluid, $\bar{p}$ is the pressure, T is the temperature inside boundary layer, $\bar{N}$ is the micro-rotation or angular velocity, j is the micro-inertia per unit mass, ρ denotes the density of the fluid, μ is the dynamic viscosity of the fluid, c_p is the specific heat at constant pressure, k is the thermal conductivity, and κ, γ are the material parameters.

1.5 Stream Function

Stream function is a very useful device in the study of fluid dynamics and was derived by the French mathematician Joseph Louis Lagrange in 1781. A stream function is defined**,** for two and three dimensional flows. The latter one is quite complicated and not necessary for our study. We restrict ourselves to two-dimensional flows.

The stream function ψ is a function of x and y. It is defined in terms of the flow velocities as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{1.12}$$

The stream function defined here satisfies the two-dimensional continuity equation. Thus, if $\psi(x, y)$ is known and is a continuously differentiable function, the velocity components u and v can be determined (White, 1998).

1.6 Porous Medium

A porous medium is composed of a persistent solid part, called solid matrix, and the remaining void space (or pore space) that can be filled with one or more fluids (e. g. water, oil and gas). Typical examples of a porous medium are soil, sand, cemented sandstone, karstic limestone, foam rubber, bread, lungs or kidneys (Bastian, 1999).

In a natural porous medium such as beach sand, sandstone, limestone, rye bread, wood, and the human lung etc., the distribution of pores with respect to shape and size is irregular. Transport properties in fluid-saturated porous media have enormous modern industrial applications in: the petroleum industry, geothermal, insulation for buildings, heat exchange between soil and atmosphere, flat plate solar collectors, flat plate condensers in refrigerators and many other areas, (Nield and Beijan, 2006; Vafai, 2010; Vadasz, 2008).

1.7 Non-dimensional Numbers

In order to get a first hand knowledge about the different phenomena occurring in heat and mass transfer problems, we should have a discussion about the fundamental dimensionless parameters which govern the process. Dimensionless parameters are normally the ratios of some forces acting on a fluid flow or the ratios of some fluid parameters involved during fluid flow under different situations, which govern the processes. Various dimensionless parameters to be considered in this study are discussed as follows:

1.7.1 Reynolds number

Reynolds number is the most important parameter of the dynamics of viscous fluid. It represents the ratio of inertia to viscous force and is defined by

$$Re = \frac{\rho \frac{U^2}{L}}{\frac{\mu U}{L^2}} = \frac{UL}{\nu}. \tag{1.13}$$

Here U, L, ρ, μ and ν represent characteristic velocity, reference length, density, dynamic viscosity and kinematic viscosity respectively. If *Re* is small the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand, if *Re* is large the inertia force will be predominant and in such case the effect of viscosity can be considered to be confined in a thin layer know as boundary layer adjacent to the surface. For large Reynolds number the flow ceases to be laminar and becomes turbulent (Borthakur and Hazarika, 2010).

1.7.2 Prandtl number

The Prandtl number is the ratio of the kinematic viscosity to the thermal diffusivity and is defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}. \tag{1.14}$$

Here α is the thermal diffusivity of the fluid, k is the thermal conductivity of the fluid and c_p is the specific heat at constant pressure of the fluid, where the value of ν shows the effect of viscosity of the fluid. For small value of ν thin region in the immediate neighborhood of the surface will be affected by the viscosity called the thermal boundary layer. The quantity $\alpha = \frac{k}{\rho c_p}$ represents thermal diffusivity due to heat conduction. For small value of α, the thin regions will be affected by heat conduction which is known as thermal boundary layer. Thus, Prandtl number shows the relative importance of heat conduction and viscosity of the fluid. It is a material property and thus varies from fluid to fluid. Liquid metals have small Prandtl number (e.g. $Pr = 0.024$ for mercury), gases are slightly less than unity (e.g. $Pr = 0.7$ for Helium), light liquids some what higher than unity and oils have very high Pr **(**Borthakur and Hazarika, 2010).

1.7.3 Grashof number

This number generally arises in the case of free convection heat transfer. It is the ratio of buoyancy force to the viscous force acting on a fluid flow. In the case of a fluid flow where the free convection of heat transfer occurs, Grashof number indicates the type of flow, whether the flow is laminar or turbulent, at higher Grashof numbers, the boundary layer is turbulent, at lower Grashof numbers, the boundary layer is laminar **(**Borthakur and Hazarika, 2010). The Grashof number is defined as

$$Gr = \frac{g \beta \Delta T L^3}{\nu^2}, \tag{1.15}$$

where g stands for the gravitational acceleration, β signifies the coefficient of volumetric change and ΔT represents the temperature difference

1.7.4 Eckert number

For incompressible flow, it determines the relative rise in temperature of the fluid through adiabatic compression. In high speed flow, it is defined as

$$Ec = \frac{U^2}{c_p \Delta T}. \tag{1.16}$$

The work of compression and that of friction become important when the characteristic velocity is comparable with or much greater than the sound or when the prescribed temperature difference is small compared to the absolute temperature of the free stream. It is important in high speed heat transfer problem and very viscous fluid. It is associated with viscous dissipation **(**Borthakur and Hazarika, 2010).

1.7.5 Nusselt number

A Nusselt number close to one, shows that convection and conduction are of similar magnitude, which is characteristic of laminar flow. A larger Nusselt number corresponds to more active convection with turbulent flow typically in the 100–1000 range. The convection and conduction heat flows are parallel to each other and to the surface normal of the boundary surface, and are all perpendicular to the mean fluid flow in the simple case (Minea, 2012*)*. The Nusselt number is defined as

$$Nu = \frac{hL}{k}, \tag{1.17}$$

where h is the convective heat transfer coefficient.

1.7.6 Rayleigh number

The Rayleigh number is associated with buoyancy driven flow (also known as free convection or natural convection). When the Rayleigh number is below the critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection (Minea, 2012*)*. The Rayleigh number is described as

$$Ra = \frac{g\,\beta \Delta T\, L^3}{\nu \alpha}. \tag{1.18}$$

1.7.7 Friction Factor

The non-dimensional shear stress at the surface is defined as the friction factor and is given by

$$C_f = \frac{\tau_w}{\rho U^2}, \tag{1.19}$$

where $\tau_w = \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0}$ is the shearing stress on the surface of the body.

We now explain the similarity solutions and group transformations method, focus on the definition of group, groups of transformations, one-parameter Lie group of transformations, scaling group of transformations and scaling method algorithm to determine similarity transformations.

1.8 Similarity Solutions and Group Transformations Method

As mentioned above the boundary layer equations can be represented by differential equations. Similarity solutions are defined mathematically as a solution where a change of variables allows for a reduction in the number of independent variables. Similarity solutions play an important role in fluid mechanics and heat transfer. A similarity ransformations reduces the governing partial differential equations to ordinary differential equations which are much easier to solve numerically.

The symmetry group transformations method is an important method to transfer partial differential equations to ordinary differential equations (similarity equations). We will adopt the group transformation method in this thesis. The method allows one to find the symmetries (infinitesimal generators of Lie group) of the differential equations which give family of equations invariant. With the symmetries of the differential equations, a solution can be obtained (Bluman and Kumei, 1989).

The group theory approach will be used in this thesis to obtain similarity transformation of the problems under investigation. A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms (Bluman and Cole, 1974; Bluman and Kumei, 1989; Hill, 1992):

(i) Closure property: For any element a and b of G, $\phi(a,b)$ is an element of G.

(ii) Associative property: For any elements a,b and c of G,

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c). \tag{1.20}$$

(iii) Identity element: There is a distinguished element e of G, called the identity element, such that for any element a in G

$$\phi(a,e) = \phi(e,a) = a. \tag{1.21}$$

(iv) Inverse element: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e. \tag{1.22}$$

1.8.1 Groups of transformations

Let $x = (x_1, x_2, ..., x_n)$ lie in region $D \subset R^n$. The set of transformations

$$x^* = X\,(x;\varepsilon), \tag{1.23}$$

defined for each x in D depending on parameter ε in set $S \subset R$, with $\varphi(\varepsilon, \delta_1)$ defining a law of composition of parameters ε and δ_1 in S, forms a group of transformations on D if the following hold (Bluman and Cole, 1974; Bluman and Kumei, 1989; Hill, 1992):

(i) For each parameter ε in S the transformations are one-to-one onto D. Hence, x^* lies in D.

(ii) S with the law of composition ϕ forms a group G.

(iii) $x^* = x$ when $\varepsilon = e$, i.e. $X\,(x, e) = x$.

(iv) If $x^* = X\,(x;\varepsilon), x^{**} = X\,(x^*;\delta_1)$, then $x^{**} = X\left(x;\phi(\varepsilon,\delta_1)\right)$.

1.8.2 One-parameter Lie group of transformations

A one-parameter group of transformations defines a one-parameter Lie group of transformations if in addition to satisfying axioms (i)-(iv) of definition (1.8.1) (Bluman and Cole, 1974; Bluman and Kumei, 1989; Hill, 1992):

(v) ε is a continuous parameter, i.e. S is an interval in R. Without loss of generality, $\varepsilon = 0$ corresponds to the identity element e.

(vi) X is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(vii) $\varphi(\varepsilon, \delta_1)$ is an analytic function of ε and $\delta_1, \varepsilon \in S$.

1.8.3 Scaling method algorithm to determine similarity transformations

We will introduce the method which will be used in this study in an algorithm form. Here, we consider the system of partial differential equations which contains p independent variables x_i and q dependent variables y_j.

Step 1: Assume that $x_i^* = e^{\varepsilon c_i} x_i$, $y_j^* = e^{\varepsilon c_{j+p}} y_j$, where $i = 1,..., p$ and $j = 1,..., q$.

Step 2: Substitute values from step 1 into the original system of partial differential equations.

Step 3: Apply the invariant condition: the resulting system should be invariant under the scaling transformations in step 1.

Step 4: Solving this linear system found in step 3 for c's.

Step 5: Formulate the characteristic equations as follows

$$\frac{dx_1}{c_1 x_1} = \frac{dx_2}{c_2 x_2} = ... = \frac{dx_p}{c_p x_p} = \frac{dy_1}{c_{p+1} y_1} = \frac{dy_2}{c_{p+2} y_2} = ... = \frac{dy_q}{c_{p+q} y_q}. \quad (1.24)$$

Step 6: Using the characteristic equations, we can find the new independent similarity variable η in terms of x_i, y_j's, also we obtain the similarity transformations.

Step7: Substituting similarity variables in the original partial system to obtain the similarity equations, we get a new system with fewer number of independent variables.

If the system contains three and more independent variables, repeat the above mentioned procedure until we get one independent variable in terms of the original independent variables. Finally we get similarity equations (Bluman and Cole, 1974; Bluman and Kumei, 1989; Hill, 1992).

Example:

To explain the scaling group transformations method, we will transform the Falkner-Skan equation to ordinary differential equation using the scaling group transformations method

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+\frac{\partial^2 u}{\partial y^2}, \tag{1.25}$$

where its boundary conditions are

$$\begin{aligned} &u=0 \quad v=0, \quad \text{at} \quad y=0, \\ &u=u_e(x), \quad \text{as} \quad y\to\infty. \end{aligned} \tag{1.26}$$

Here $u_e(x)=x^m$ is the velocity of the free stream, m is the Falkner-Skan power law parameter. We introduce the stream function ψ which is defined as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$ to reduce the number of equations and number of dependent variables. Then Eq. (1.25) becomes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=u_e\frac{du_e}{dx}+\frac{\partial^3\psi}{\partial y^3}, \tag{1.27}$$

and the boundary conditions (1.26) become

$$\begin{aligned} &\frac{\partial\psi}{\partial y}=0,\ \frac{\partial\psi}{\partial x}=0 \quad \text{at} \quad y=0, \\ &\frac{\partial\psi}{\partial y}=u_e(x) \quad \text{as} \quad y\to\infty. \end{aligned} \tag{1.28}$$

We transform the partial differential equation (1.27) and the boundary conditions (1.28) to an ordinary differential equation using scaling group transformations

$$\Gamma : x^* = e^{\varepsilon c_1}x,\ y^* = e^{\varepsilon c_2}y,\ \psi^* = e^{\varepsilon c_3}\psi \tag{1.29}$$

Here ε is the parameter of the group Γ and c_i's , $(i=1,2,3)$ are arbitrary real numbers. The Eqs. (1.27) and (1.28) will remain invariant under the group transformations in Eq. (1.29) if the following relationships hold:

$$e^{\varepsilon(c_1+2c_2-2c_3)}\left(\frac{\partial\psi^*}{\partial y^*}\frac{\partial^2\psi^*}{\partial y^*\partial x^*}-\frac{\partial\psi^*}{\partial x^*}\frac{\partial^2\psi^*}{\partial\left(y^*\right)^2}\right)=$$
$$me^{-\varepsilon c_1(2m-1)}\left(x^*\right)^{2m-1}+e^{\varepsilon(3c_2-c_3)}\frac{\partial^3\psi^*}{\left(y^*\right)^3}, \tag{1.30}$$

$$e^{\varepsilon(c_2-c_3)}\frac{\partial\psi^*}{\partial y^*}=0,\ e^{\varepsilon(c_1-c_3)}\frac{\partial\psi^*}{\partial x^*}=0 \quad \text{at} \quad y^*=0,$$
$$e^{\varepsilon(c_2-c_3)}\frac{\partial\psi^*}{\partial y^*}=u_e\left(x^*\right) \quad \text{as} \quad y^*\to\infty. \tag{1.31}$$

Equating powers of e, we have

$$c_1+2c_2-2c_3=-c_1\left(2m-1\right)=3c_2-c_3 \tag{1.32}$$

Solving the Eq. (1.32), we have the following relationship among the exponents

$$c_2=\frac{1}{2}(1-m)c_1,\ \ c_3=\frac{1}{2}(1+m)c_1. \tag{1.33}$$

the characteristic equations are

$$\frac{dx}{x}=\frac{dy}{\frac{1}{2}(1-m)y}=\frac{d\psi}{\frac{1}{2}(1+m)\psi}. \tag{1.34}$$

Using the characteristic Eqs. (1.34), we find these equations.

$$\frac{dx}{x}=\frac{dy}{\frac{1}{2}(1-m)y},\quad \frac{dx}{x}=\frac{d\psi}{\frac{1}{2}(1+m)\psi}. \tag{1.35}$$

Solving these equations by using the integration we obtain,

$$\eta=x^{\frac{m-1}{2}}y\,,\psi=x^{\frac{m+1}{2}}f\ \left(\eta\right). \tag{1.36}$$

Here η and $f\left(\eta\right)$ are similarity independent and dependent variables recpectively. Substituting Eq. (1.36) into Eq. (1.27) and boundary conditions (1.28), we get

$$f''' + \frac{m+1}{2} ff'' + m\left(1 - f'^2\right) = 0, \tag{1.37}$$

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{1.38}$$

1.9 Runge-Kutta-Fehlberg Method

The Runge-Kutta method is a numerical technique to solve an initial value problem of the form

$$\frac{dy}{dx} = f\ (x, y), \quad y\ (x_0) = y_0. \tag{1.39}$$

The most popular Runge-Kutta method is the classical Runge-Kutta fourth order method. One way to guarantee accuracy in the solution of an initial valued problem is to solve the problem twice using step sizes h and $h/2$ and compare answers at the mesh points corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough (Mathews and Fink, 2004).

The Runge-Kutta-Fehlberg method (RKF45) is a technique to resolve this problem. It is called RKF45 because the fourth-order method with five stages is used together with a fifth-order method with six stages, that uses all of the points of the first one. It has a procedure to determine if the proper step size h is being used. At each step, two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the two answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size is increased. Each step requires the use of the following six values (Mathews and Fink, 2004).

First, we need the definitions of the following;

$$\begin{aligned}
K_1 &= hf\ (x_k, y_k),\\
K_2 &= hf\ \left(x_k + \frac{1}{4}h, y_k + \frac{1}{4}K_1\right),\\
K_3 &= hf\ \left(x_k + \frac{3}{8}h, y_k + \frac{3}{32}K_1 + \frac{9}{32}K_2\right),\\
K_4 &= hf\ \left(x_k + \frac{12}{13}h, y_k + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 + \frac{7296}{2197}K_3\right),\\
K_5 &= hf\ \left(x_k + h, y_k + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4\right),\\
K_6 &= hf\ \left(x_k + \frac{1}{2}h, y_k - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5\right).
\end{aligned} \tag{1.40}$$

Then an approximation to the solution of the initial value problem (I.V.P) is made using a Runge-Kutta method of order 4:

$$y_{k+1} = y_k + \frac{25}{216}K_1 + \frac{1408}{2565}K_3 + \frac{2197}{4101}K_4 - \frac{1}{5}K_5, \tag{1.41}$$

where the four function values K_1, K_3, K_4 and K_5 are used. A better value for the solution is determined using a Runge-Kutta method of order 5:

$$z_{k+1} = y_k + \frac{16}{135}K_1 + \frac{6656}{12825}K_3 + \frac{28561}{56430}K_4 - \frac{9}{50}K_5 + \frac{2}{55}K_6 \tag{1.42}$$

The optimal step size sh can be determined by multiplying the scalar s times the current step size h. The scalar s is

$$s = \left(\frac{\text{tol } h}{2|z_{k+1} - y_{k+1}|}\right)^{\frac{1}{4}} \approx 0.84\left(\frac{\text{tol } h}{|z_{k+1} - y_{k+1}|}\right)^{\frac{1}{4}} \tag{1.43}$$

1.10 Background and Motivation

A power-law fluid model, or the Ostwald-de Waele relationship, is a type of generalized Newtonian fluid for which the relationship between shear stress and