DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

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DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

by

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TABLE OF CONTENTS

SYMBOLS

Symbol	Description	Page
$\mathcal{A}_{p,n}$	Class of all analytic functions f of the form $f(z) = z^p + a_{n+p}z^{n+p} + a_{n+p+1}z^{n+p+1} + \dots \quad (z \in \mathbb{D})$	99
$\mathcal{A}_p$	Class of all p-valent analytic functions f of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{D})$	2
$\mathcal{A}_{n,b}$	Class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$ where b is a fixed non-negative real number.	60
$\mathcal{A}$	Class of analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D})$	1
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	4
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\}$	8
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	6
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$	1
$\mathbb{D}^*$	Open punctured unit disk $\{z \in \mathbb{C} : 0 < \|z\| < 1\}$	6
$\partial\mathbb{D}$	Boundary of unit disk $\mathbb{D}$	9
$\mathcal{G}_\alpha(\varphi)$	$\left\{f \in \mathcal{A} : (1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\}$	54
$\mathcal{H}(\mathbb{D})$	Class of all analytic functions in $\mathbb{D}$	1
$\mathcal{H}[a,n] = \mathcal{H}_n(a)$	Class of all analytic functions f in $\mathbb{D}$ of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$	1

$\mathcal{H} := \mathcal{H}[1,1]$	Class of analytic functions f in $\mathbb{D}$ of the form $f(z) = 1 + a_1 z + a_2 z^2 + \cdots$	1
$\mathcal{H}_{\mu,n}$	Class of analytic functions p on $\mathbb{D}$ of the form $p(z) = 1 + \mu z^n + p_{n+1} z^{n+1} + \cdots$	60
$\mathcal{H}_{\mathcal{SB}}(\varphi)$	$\{f \in \mathcal{SB} : f'(z) \prec \varphi(z)$ and $g'(w) \prec \varphi(w),\ g(w) := f^{-1}(w)\}$	34
$H_q(n)$	Hankel determinants of functions $f \in \mathcal{A}$	36
$\mathcal{L}(\alpha, \varphi)$	$\left\{f \in \mathcal{S} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z)\right\}$	13
$\mathcal{M}(\alpha, \varphi)$	$\left\{f \in \mathcal{S} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\}$	13
$\mathcal{P}$	$\{p \in \mathcal{H} : \text{with } \operatorname{Re} p(z) > 0,\ z \in \mathbb{D}\}$	6
$\mathcal{P}(\alpha)$	$\{p \in \mathcal{H} : \text{with } \operatorname{Re} p(z) > \alpha,\ z \in \mathbb{D}\}$	6
$\mathbb{R}$	Set of all real numbers	2
Re	Real part of a complex number	4
$\mathcal{R}^{\tau}_{\gamma}(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \varphi(z)\right\}$	49
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$	2
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	5
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}(\varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$	8
$\mathcal{ST}(\alpha, \varphi)$	$\left\{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z)\right\}$	13
Σ	Class of all meromorphic functions f of the form $f(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$	6
$\Sigma_{n,b}$	Class of all meromorphic functions f of the form	73

	$f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \cdots \quad (b \leq 0)$	
$\prec$	Subordinate to	6
$\mathcal{SB}$	Class of bi-univalent functions	12
$\Psi_n[\Omega, q]$	Class of admissible functions for differential subordination	9
$\Psi_{\mu,n}[\Omega]$	Class of admissible functions for fixed second coefficient	60

SUBORDINASI PEMBEZA DAN MASALAH PEKALI UNTUK FUNGSI-FUNGSI ANALISIS

ABSTRAK

Lambangkan $\mathcal{A}$ sebagai kelas fungsi analisis ternormal pada cakera unit $\mathbb{D}$ berbentuk $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Fungsi f dalam $\mathcal{A}$ adalah univalen jika fungsi tersebut ialah pemetaan satu ke satu. Tesis ini mengkaji lima masalah penyelidikan.

Fungsi $f \in \mathcal{A}$ dikatakan dwi univalen dalam $\mathbb{D}$ jika kedua-dua fungsi f dan songsangannya f^{-1} adalah univalen dalam $\mathbb{D}$. Anggaran pekali awal, $|a_2|$ dan $|a_3|$, fungsi dwi univalen akan dikaji untuk f dan f^{-1} yang masing-masing terkandung di dalam subkelas fungsi univalen tertentu. Seterusnya, batas penentu Hankel kedua $H_2(2) = a_2 a_4 - a_3^2$ untuk fungsi analisis f dengan $zf'(z)/f(z)$ dan $1 + zf''(z)/f'(z)$ subordinat kepada suatu fungsi analisis tertentu diperoleh.

Bermotivasikan kerja terdahulu dalam subordinasi pembeza peringkat kedua untuk fungsi analisis dengan pekali awal tetap, syarat cukup bak-bintang dan univalen untuk suatu subkelas fungsi berpekali kedua tetap ditentukan. Kemudian, syarat cukup cembung untuk fungsi yang pekali keduanya tidak ditetapkan dan yang memenuhi ketaksamaan pembeza peringkat kedua dan ketiga tertentu diperoleh.

Akhir sekali, subkelas fungsi multivalen yang memenuhi syarat bak-bintang dan hampir cembung dikaji.

Beberapa aspek permasalahan dalam teori fungsi univalen dibincangkan dalam tesis ini dan hasil-hasil menarik diperoleh.

DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

ABSTRACT

Let $\mathcal{A}$ be the class of normalized analytic functions f on the unit disk $\mathbb{D}$, in the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. A function f in $\mathcal{A}$ is univalent if it is a one-to-one mapping. This thesis discussed five research problems.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Estimates on the initial coefficients, $|a_2|$ and $|a_3|$, of bi-univalent functions f are investigated when f and f^{-1} respectively belong to some subclasses of univalent functions. Next, the bounds for the second Hankel determinant $H_2(2) = a_2 a_4 - a_3^2$ of analytic function f for which $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ is subordinate to certain analytic function are obtained.

Motivated by the earlier work on second order differential subordination for analytic functions with fixed initial coefficient, the sufficient conditions for starlikeness and univalence for a subclass of functions with fixed second coefficient are obtained. Then, without fixing the second coefficient, the sufficient condition for convexity of these functions satisfying certain second order and third order differential inequalities are determined.

Lastly, the close-to-convexity and starlikeness of a subclass of multivalent functions are investigated.

A few aspects of problems in univalent function theory is discussed in this thesis and some interesting results are obtained.

CHAPTER 1

INTRODUCTION

1.1 Univalent function

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. A function f is analytic at a point $z_0 \in D$ if it is differentiable in some neighborhood of z_0 and it is analytic in a domain D if it is analytic at all points in domain D. An analytic function f is said to be univalent in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. An analytic function f is locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0. The well known Riemann Mapping Theorem states that every simply connected domain (which is not the whole complex plane $\mathbb{C}$), can be mapped conformally onto the unit disk $\mathbb{D}$.

Theorem 1.1 (Riemann Mapping Theorem) [29, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathbb{D}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

In view of this theorem, the study of analytic univalent functions on a simply connected domain can be restricted to the open unit disk $\mathbb{D}$.

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions defined on $\mathbb{D}$. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with $\mathcal{H} \equiv \mathcal{H}[1, 1]$.

Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $\mathbb{D}$, and normalized by $f(0) = 0$, and $f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

For a fixed $p \in \mathbb{N} := \{1, 2, \dots\}$, let $\mathcal{A}_p$ be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

that are p-valent (multivalent) in the open unit disk, with $\mathcal{A} := \mathcal{A}_1$.

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The function k given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathbb{D})$$

is called the Koebe function, which maps $\mathbb{D}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$, and is univalent. It plays a very important role in the study of the class $\mathcal{S}$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, for $\beta \in \mathbb{R}$, are the only extremal functions for various problem in the class $\mathcal{S}$. In 1916, Bieberbach [19] conjectured that for $f \in \mathcal{S}$, $|a_n| \leq n$, $(n \geq 2)$. He proved only for the case when $n = 2$.

Theorem 1.2 (Bieberbach's Conjecture) [19] *If $f \in \mathcal{S}$, then $|a_n| \leq 2$ $(n \geq 2)$ with equality if and only if f is the rotation of the Koebe function k.*

For the cases $n = 3$, and $n = 4$ the conjecture was proved by Lowner [58] and Garabedian and Schiffer [34], respectively. Later, Pederson and Schiffer [98] proved the conjecture for $n = 5$, and for $n = 6$, it was proved by Pederson [97] and Ozawa [95], independently. In 1985, Louis de Branges [20], proved the Bieberbach's conjecture for all the coefficients n.

Theorem 1.3 (de Branges Theorem or Bieberbach's Theorem) [20] *If* $f \in \mathcal{S}$, *then*

$$|a_n| \leq n \quad (n \geq 2),$$

with equality if and only if f *is the Koebe function* k *or one of its rotations.*

Bieberbach's theorem has many important properties in univalent functions. These include the well known covering theorem: If $f \in \mathcal{S}$, then the image of $\mathbb{D}$ under f contains a disk of radius 1/4.

Theorem 1.4 (Koebe One-Quarter Theorem) [29, p. 31] *The range of every function* $f \in \mathcal{S}$ *contains the disk* $\{w \in \mathbb{C} : |w| < 1/4\}$.

The Distortion theorem, being another consequence of the Bieberbach theorem gives sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.5 (Distortion Theorem) [29, p. 32] *For each* $f \in \mathcal{S}$,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1).$$

The distortion theorem can be used to obtain sharp upper and lower bounds for $|f(z)|$ which is known as the Growth theorem.

Theorem 1.6 (Growth Theorem) [29, p. 33] *For each* $f \in \mathcal{S}$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1).$$

Another consequence of the Bieberbach theorem is the Rotation theorem.

Theorem 1.7 (Rotation Theorem) [29, p. 99] *For each* $f \in \mathcal{S}$,

$$|\arg f'(z)| \leq \begin{cases} 4sin^{-1}r, & r \leq \frac{1}{\sqrt{2}}, \\ \pi + \log \frac{r^2}{1-r^2}, & r \geq \frac{1}{\sqrt{2}}, \end{cases}$$

where $|z| = r < 1$. *The bound is sharp.*

The Fekete-Szego coefficient functional also arises in the investigation of univalency of analytic functions.

Theorem 1.8 (Fekete-Szego Theorem) [29, p. 104] *For each* $f \in \mathcal{S}$,

$$|a_3 - \alpha a_2^2| \leq 1 + 2e^{-2\alpha/(1-\alpha)}, \quad (0 < \alpha < 1).$$

1.2 Subclasses of univalent functions

The long gap between the Bieberbach's conjecture in 1916 and its proof by de Branges in 1985 motivated researchers to consider classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions.

A set D in the complex plane is called *convex* if for every pair of points w_1 and w_2 lying in the interior of D, the line segment joining w_1 and w_2 also lies in the interior of D, i.e.

$$tw_1 + (1-t)w_2 \in D \quad \text{for} \quad 0 \leq t \leq 1.$$

If a function $f \in \mathcal{A}$ maps $\mathbb{D}$ onto a convex domain, then f is a *convex function.* The class of all convex functions in $\mathcal{A}$ is denoted by $\mathcal{CV}$. An analytic description of the class $\mathcal{CV}$ is given by

$$\mathcal{CV} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

Let w_0 be an interior point of D. A set D in the complex plane is called *starlike* with respect to w_0 if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D, i.e.

$$(1-t)w + tw_0 \ \in D \quad \text{for} \quad 0 \le t \le 1.$$

If a function $f \in \mathcal{A}$ maps $\mathbb{D}$ onto a starlike domain, then f is a *starlike function*. The class of starlike functions with respect to origin is denoted by $\mathcal{ST}$. Analytically,

$$\mathcal{ST} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0 \right\}.$$

In 1936, Robertson [105] introduced the concepts of convex functions of order α and starlike functions of order α for $0 \le \alpha < 1$. A function $f \in \mathcal{A}$ is said to be *convex of order* α if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{D}),$$

and *starlike of order* α if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{D}).$$

These classes are respectively denoted by $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$.

Note that $\mathcal{CV}(0) = \mathcal{CV}$ and $\mathcal{ST}(0) = \mathcal{ST}$. An important relationship between convex and starlike functions was first observed by Alexander [1] in 1915 and known later as Alexander's theorem.

Theorem 1.9 (Alexander's Theorem) [29, p. 43] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$.*

From this, it is easily proven that $f \in \mathcal{CV}(\alpha)$ if and only if $zf' \in \mathcal{ST}(\alpha)$.

Another subclass of $\mathcal{S}$ that has an important role in the study of univalent functions is the class of close-to-convex functions introduced by Kaplan [45] in 1952. A function $f \in \mathcal{A}$ is *close-to-convex* in $\mathbb{D}$ if there is a convex function g and a real number θ, $-\pi/2 < \theta < \pi/2$, such that

$$\mathrm{Re}\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

The class of all such functions is denoted by $\mathcal{CCV}$. The subclasses of $\mathcal{S}$, namely convex, starlike and close-to-convex functions are related as follows:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV} \subset \mathcal{S}.$$

The well known Noshiro-Warschawski theorem states that a function $f \in \mathcal{A}$ with positive derivative in $\mathbb{D}$ is univalent.

Theorem 1.10 [82, 131] *For some real α, if a function f is analytic in a convex domain D and*

$$\mathrm{Re}\left(e^{i\alpha}f'(z)\right) > 0,$$

then f is univalent in D.

Kaplan [45] applied Noshiro-Warschawski theorem to prove that every close-to-convex function is univalent.

The class of meromorphic functions is yet another subclass of univalent functions. Let Σ denote the class of normalized *meromorphic* functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

that are analytic in the punctured unit disk $\mathbb{D}^* := \{z : 0 < |z| < 1\}$ except for a simple pole at 0.

A function f is said to be subordinate to F in $\mathbb{D}$, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in $\mathbb{D}$, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

Let $\mathcal{P}$ be the class of all analytic functions p of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}). \tag{1.1}$$

Any function in $\mathcal{P}$ is called a function with positive real part, also known as Caratheodory function. The following lemma is known for functions in $\mathcal{P}$.

Lemma 1.1 [29] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots ,$$

then the following sharp estimate holds:

$$|p_n| \leq 2 \quad (n = 1, 2, \ldots).$$

The above fact will be used often in the thesis especially in Chapters 2 and 3. More generally, for $0 \leq \alpha < 1$, we denote by $\mathcal{P}(\alpha)$ the class of analytic functions $p \in \mathcal{P}$ with

$$\operatorname{Re} p(z) > \alpha \quad (z \in \mathbb{D}).$$

In terms of subordination, the analytic condition (1.1) can be written as

$$p(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}).$$

This follows since the mapping $q(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane.

Ma and Minda [59] have given a unified treatment of various subclasses consisting of starlike and convex functions by replacing the superordinate function

$q(z) = (1+z)/(1-z)$ by a more general analytic function. For this purpose, they considered an analytic function φ with positive real part on $\mathbb{D}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. The class of Ma-Minda starlike functions denoted by $\mathcal{ST}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z)$$

and similarly the class of Ma-Minda convex functions denoted by $\mathcal{CV}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad (z \in \mathbb{D}).$$

respectively.

1.3 Differential subordination

Recall that a function f is said to be subordinate to F in $\mathbb{D}$, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in $\mathbb{D}$, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

The basic definitions and theorems in the theory of subordination and certain applications of differential subordinations are stated in this section. The theory of differential subordination were developed by Miller and Mocanu [61].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the second order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.2}$$

then p is called a *solution of the differential subordination.* The univalent function q is called a *dominant* of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all p satisfying (1.2). A dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (1.2) is said to be the *best dominant* of (1.2). The best dominant is unique up to a rotation of $\mathbb{D}$.

If $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)*-solution*, q an (a, n)*-dominant,* and q_1 the *best* (a, n)*-dominant.* Let $\Omega \subset \mathbb{C}$ and let (1.2) be replaced by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \text{ for all } z \in \mathbb{D}, \tag{1.3}$$

where Ω is a simply connected domain containing $h(\mathbb{D})$. Even though this is a differential inclusion and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in $\mathbb{D}$, the condition in (1.3) shall also be referred as a *second order differential subordination,* and the same definition of solution, dominant and best dominant as given above can be extended to this generalization. The monograph [61] by Milller and Mocanu provides more detailed information on the theory of differential subordination.

Denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathbb{D}} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} q(z) = \infty\}$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \backslash E(q)$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Miller and Mocanu [61].

Definition 1.1 [61, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and m be a positive integer. The class of admissible functions $\Psi_m[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever*

$r = q(\zeta), s = k\zeta q'(\zeta)$ *and*

$$\operatorname{Re}\left(\frac{t}{s}+1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

$z \in \mathbb{D}, \zeta \in \partial\mathbb{D}\backslash E(q)$ *and* $k \geq m$. *In particular,* $\Psi[\Omega, q] := \Psi_1[\Omega, q]$.

The next theorem is the foundation result in the theory of first and second-order differential subordinations.

Theorem 1.11 [61, Theorem 2.3b, p. 28] *Let* $\psi \in \Psi_m[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega,$$

then $p \prec q$.

1.4 Scope of thesis

This thesis will discuss five research problems. In Chapter 2, estimates on the initial coefficients for bi-univalent functions f in the open unit disk with f and its inverse $g = f^{-1}$ satisfying the conditions that $zf'(z)/f(z)$ and $zg'(z)/g(z)$ are both subordinate to a univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

In Chapter 3, the bounds for the second Hankel determinant $a_2a_4 - a_3^2$ of analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ for which either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to certain analytic function are investigated. The problem is also investigated for two other related classes defined by subordination. The classes introduced by subordination naturally include several well known classes of univalent functions and the results for some of these special classes are indicated. In particular, the estimates for the Hankel determinant of strongly starlike, parabolic starlike, lemniscate starlike functions are obtained.

In Chapter 4, several well known results for subclasses of univalent functions was extended to functions with fixed initial coefficient by using the theory of differential subordination. Further applications of this subordination theory is given. In particular, several sufficient conditions related to starlikeness, meromorphic starlikeness and univalence of normalized analytic functions are derived.

In Chapter 5, the convexity conditions for analytic functions defined in the open unit disk satisfying certain second-order and third-order differential inequalities are obtained. As a consequence, conditions are also determined for convexity of functions defined by following integral operators

$$f(z) = \int_0^1 \int_0^1 W(r,s,z)drds, \quad \text{and} \quad f(z) = \int_0^1 \int_0^1 \int_0^1 W(r,s,t,z)drdsdt.$$

In the final chapter, several sufficient conditions for close-to-convexity and starlikeness of a subclass of multivalent functions are investigated. Relevant connections with previously known results are indicated.

CHAPTER 2

COEFFICIENTS FOR BI-UNIVALENT FUNCTIONS

2.1 Introduction and preliminaries

For functions $f \in \mathcal{S}$, let f^{-1} be its inverse function. The Koebe one-quarter theorem (Theorem 1.4) ensures the existence of f^{-1}, that is, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4)\,.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let $\mathcal{SB}$ denote the class of bi-univalent functions defined in $\mathbb{D}$. Examples of functions in the class $\mathcal{SB}$ are $z/(1-z)$ and $-\log(1-z)$.

In 1967, Lewin [51] introduced this class $\mathcal{SB}$ and proved that the bound for the second coefficients of every $f \in \mathcal{SB}$ satisfies the inequality $|a_2| \leq 1.51$. He also investigated $\mathcal{SB}_1 \subset \mathcal{SB}$, the class of all functions $f = \phi \circ \psi^{-1}$, where ϕ and ψ map $\mathbb{D}$ onto domains containing $\mathbb{D}$ and $\phi'(0) = \psi'(0)$. For an example that shows $\mathcal{SB} \neq \mathcal{SB}_1$, see [23]. In 1969, Suffridge [122] showed that a function in $\mathcal{SB}_1$ satisfies $a_2 = 4/3$ and thus conjectured that $|a_2| \leq 4/3$ for all functions in $\mathcal{SB}$. Netanyahu [69], in the same year, proved this conjecture for a subclass of $\mathcal{SB}_1$. In 1981, Styer and Wright [121] showed that $a_2 > 4/3$ for some function in $\mathcal{SB}$, thus disproved the conjecture of Suffridge. For bi-univalent polynomial $f(z) = z + a_2z^2 + a_3z^3$ with real coefficients, Smith [114] showed that $|a_2| \leq 2/\sqrt{27}$ and $|a_3| \leq 4/27$ and the latter inequality being the best possible. He also showed that if $z + a_nz^n$ is bi-univalent, then $|a_n| \leq (n-1)^{n-1}/n^n$ with equality best possible for $n = 2, 3$. Kędzierawski and Waniurski [47] proved the conjecture of Smith [114] for $n = 3, 4$ in the case of bi-univalent polynomial of degree n. Extending the results of Srivasta *et al.* [118], Frasin and Aouf [33] obtained estimate of $|a_2|$ and $|a_3|$ for bi-univalent function f for which

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z) \quad \text{and} \quad (1-\lambda)\frac{g(w)}{w}+\lambda g'(w) \quad (g=f^{-1})$$

belongs to a sector in the half plane. Tan [125] improved Lewin's result to $|a_2| \leq 1.485$. For $0 \leq \alpha < 1$, a function $f \in \mathcal{SB}$ is bi-starlike of order α or bi-convex of order α if both f and f^{-1} are respectively starlike or convex of order α. These classes were introduced by Brannan and Taha [22]. They obtained estimates on the initial coefficients for functions in these classes. For some open problems and survey, see [35,115]. Bounds for the initial coefficients of several classes of functions were also investigated in [7,8,24–26,33,39,48,60,64,67,108,117–120,126,133,134].

2.2 Kędzierawski type results

In 1985, Kędzierawski [46] considered functions f belonging to certain subclasses of univalent functions while its inverse f^{-1} belongs to some other subclasses of univalent functions. Among other results, he obtained the following.

Theorem 2.1 [46] *Let $f \in \mathcal{SB}$ with Taylor series $f(z) = z + a_2 z^2 + \cdots$ and $g = f^{-1}$. Then*

$$|a_2| \leq \begin{cases} 1.5894 & \text{if } f \in \mathcal{S},\ g \in \mathcal{S}, \\ \sqrt{2} & \text{if } f \in \mathcal{ST},\ g \in \mathcal{ST}, \\ 1.507 & \text{if } f \in \mathcal{ST},\ g \in \mathcal{S}, \\ 1.224 & \text{if } f \in \mathcal{CV},\ g \in \mathcal{S}. \end{cases}$$

Consider the following classes investigated in [7,8,14].

Definition 2.1 *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. For $\alpha \geq 0$, let*

$$\mathcal{M}(\alpha,\varphi) := \left\{ f \in \mathcal{S} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

$$\mathcal{L}(\alpha,\varphi) := \left\{ f \in \mathcal{S} \, : \, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$
$$\mathcal{ST}(\alpha,\varphi) := \left\{ f \in \mathcal{S} \, : \, \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}.$$

Suppose that f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

then it is known that $g = f^{-1}$ has the expression

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots .$$

Motivated by Theorem 2.1, we will consider the following cases and then will obtain the estimates for the second and third coefficients of functions f:

1. $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{ST}(\beta,\psi)$, or $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$,

2. $f \in \mathcal{M}(\alpha,\varphi)$ and $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$,

3. $f \in \mathcal{L}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$,

where φ and ψ are analytic functions of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0) \tag{2.2}$$

and

$$\psi(z) = 1 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots, \quad (D_1 > 0). \tag{2.3}$$

2.3 Second and third coefficients of functions f when $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{ST}(\beta,\psi)$, or $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$

We begin with the cases for $f \in \mathcal{ST}(\alpha,\varphi)$.

Theorem 2.2 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{ST}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1+3\beta) + D_1(1+3\alpha)}}{\sqrt{|\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}} \tag{2.4}$$

and

$$2\rho|a_3| \leq B_1(3+10\beta) + D_1(1+2\alpha) + (3+10\beta)|B_2 - B_1| + \frac{(1+2\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \tag{2.5}$$

where $\rho := 2 + 7\alpha + 7\beta + 24\alpha\beta$.

Proof. Since $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{ST}(\beta, \psi)$, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = \psi(v(w)). \tag{2.6}$$

Define the functions p_1 and p_2 by

$$p_1(z) := \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

or, equivalently,

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right) \tag{2.7}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \cdots\right). \tag{2.8}$$

Then p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have positive real part in $\mathbb{D}$, and thus $|b_i| \leq 2$ and $|c_i| \leq 2$

(Lemma 1.1). In view of (2.6), (2.7) and (2.8), it is clear that

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)} = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad \text{and} \quad \frac{wg'(w)}{g(w)}+\frac{\beta w^2 g''(w)}{g(w)} = \psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.9}$$

Using (2.7) and (2.8) together with (2.2) and (2.3), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots \tag{2.10}$$

and

$$\psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1+\frac{1}{2}D_1b_1w+\left(\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2\right)w^2+\cdots. \tag{2.11}$$

Since

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)} = 1+a_2(1+2\alpha)z+\Big(2(1+3\alpha)a_3-(1+2\alpha)a_2^2\Big)z^2+\cdots$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\beta w^2 g''(w)}{g(w)} = 1-(1+2\beta)a_2w+\Big((3+10\beta)a_2^2-2(1+3\beta)a_3\Big)w^2+\cdots,$$

it follows from (2.9), (2.10) and (2.11) that

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \tag{2.12}$$

$$2(1+3\alpha)a_3-(1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.13}$$

$$-(1+2\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.14}$$

and

$$(3+10\beta)a_2^2 - 2(1+3\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.15}$$

It follows from (2.12) and (2.14) that

$$b_1 = -\frac{B_1(1+2\beta)}{D_1(1+2\alpha)}c_1. \tag{2.16}$$

Multiplying (2.13) with $(1+3\beta)$ and (2.15) with $(1+3\alpha)$, and adding the results give

$$a_2^2((1+3\alpha)(3+10\beta) - (1+3\beta)(1+2\alpha)) = \frac{1}{2}B_1(1+3\beta)c_2 + \frac{1}{2}D_1(1+3\alpha)b_2$$
$$+ \frac{1}{4}c_1^2(1+3\beta)(B_2 - B_1) + \frac{1}{4}b_1^2(1+3\alpha)(D_2 - D_1).$$

Substituting c_1 from (2.12) and b_1 from (2.16) in the above equation give

$$\begin{aligned} & a_2^2((1+3\alpha)(3+10\beta) - (1+3\beta)(1+2\alpha)) \\ & \quad - a_2^2\left(\frac{(1+3\beta)(1+2\alpha)^2(B_2 - B_1)}{B_1^2} + \frac{(1+2\beta)^2(1+3\alpha)(D_2 - D_1)}{D_1^2}\right) \\ & = \frac{1}{2}B_1(1+3\beta)c_2 + \frac{1}{2}D_1(1+3\alpha)b_2 \end{aligned}$$

which lead to

$$a_2^2 = \frac{B_1^2 D_1^2[B_1(1+3\beta)c_2 + D_1(1+3\alpha)b_2]}{2[\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2]},$$

where $\rho := 2 + 7\alpha + 7\beta + 24\alpha\beta$, which, in view of $|b_2| \leq 2$ and $|c_2| \leq 2$, gives us the desired estimate on $|a_2|$ as asserted in (2.4).

Multiplying (2.13) with $(3+10\beta)$ and (2.15) with $(1+2\alpha)$, and adding the results give

$$2((1+3\alpha)(3+10\beta)-(1+3\beta)(1+2\alpha))a_3 = \frac{1}{2}B_1(3+10\beta)c_2 + \frac{1}{2}D_1(1+2\alpha)b_2 + \frac{c_1^2}{4}(3+10\beta)(B_2-B_1) + \frac{b_1^2}{4}(D_2-D_1)(1+2\alpha).$$

Substituting b_1 from (2.16) in the above equation lead to

$$2\rho a_3 = \frac{1}{2}\left[B_1(3+10\beta)c_2 + D_1(1+2\alpha)b_2\right] + \frac{c_1^2}{4}\left[(3+10\beta)(B_2-B_1) + \frac{(1+2\beta)^2 B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],$$

and this yields the estimate given in (2.5). ∎

Remark 2.1 *When* $\alpha=\beta=0$ *and* $B_1=B_2=2$, $D_1=D_2=2$, *inequality* (2.4) *reduces to the second result in Theorem 2.1.*

In the case when $\beta=\alpha$ and $\psi=\varphi$, Theorem 2.2 reduces to the following corollary.

Corollary 2.1 *Let* f *given by* (2.1) *and* $g=f^{-1}$. *If* $f,g\in\mathcal{ST}(\alpha,\varphi)$, *then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2(1+4\alpha)+(B_1-B_2)(1+2\alpha)^2|}}, \tag{2.17}$$

and

$$|a_3| \le \frac{B_1+|B_2-B_1|}{(1+4\alpha)}. \tag{2.18}$$

For φ given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1+2\gamma z+2\gamma^2 z^2+\cdots \quad (0<\gamma\le 1),$$

we have $B_1=2\gamma$ and $B_2=2\gamma^2$. Hence, when $\alpha=0$, the inequality (2.17) reduces to the following result.

Corollary 2.2 [22, Theorem 2.1] *Let f given by* (2.1) *be in the class of strongly bi-starlike functions of order γ, $0 < \gamma \le 1$. Then*

$$|a_2| \le \frac{2\gamma}{\sqrt{1+\gamma}}.$$

On the other hand, when $\alpha = 0$ and

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots$$

so that $B_1 = B_2 = 2(1-\beta)$, the inequalities in (2.17) and (2.18) reduce to the following result.

Corollary 2.3 [22, Theorem 3.1] *Let f given by* (2.1) *be in the class of bi-starlike functions of order β, $0 < \beta \le 1$. Then*

$$|a_2| \le \sqrt{2(1-\beta)} \quad \textit{and} \quad |a_3| \le 2(1-\beta).$$

Theorem 2.3 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, then*

$$|a_2| \le \frac{B_1 D_1 \sqrt{B_1(1+2\beta) + D_1(1+3\alpha)}}{\sqrt{|\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2 - B_1)D_1^2 - (1+\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}} \tag{2.19}$$

and

$$2\rho|a_3| \le B_1(3+5\beta) + D_1(1+2\alpha) + (3+5\beta)|B_2 - B_1| + \frac{(1+\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \tag{2.20}$$

where $\rho := 2 + 7\alpha + 3\beta + 11\alpha\beta$.

Proof. Let $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, $g = f^{-1}$. Then there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad (1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = \psi(v(w)), \tag{2.21}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1-(1+\beta)a_2w + ((3+5\beta)a_2^2 - 2(1+2\beta)a_3)w^2 + \cdots,$$

equations (2.10), (2.11) and (2.21) yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \tag{2.22}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.23}$$

$$-(1+\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.24}$$

and

$$(3+5\beta)a_2^2 - 2(1+2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \tag{2.25}$$

It follows from (2.22) and (2.24) that

$$b_1 = -\frac{B_1(1+\beta)}{D_1(1+2\alpha)}c_1. \tag{2.26}$$

Multiplying (2.23) with $(1+2\alpha)$ and (2.25) with $(1+3\alpha)$, and adding the results give

$$a_2^2(2+7\alpha+3\beta+11\alpha\beta) = \frac{B_1}{2}(1+2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$
$$+\frac{c_1^2}{4}(1+2\beta)(B_2-B_1) + \frac{b_1^2}{4}(1+3\alpha)(D_2-D_1)$$

Substituting c_1 from (2.22) and b_1 from (2.26) in the above equation give

$$a_2^2(2+7\alpha+3\beta+11\alpha\beta)$$
$$-\frac{a_2^2(1+2\alpha)^2}{B_1^2}\left((1+2\beta)(B_2-B_1) + \frac{(1+3\alpha)(D_2-D_1)(1+\beta)^2B_1^2}{(1+2\alpha)^2D_1^2}\right)$$
$$= \frac{B_1}{2}(1+2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$

which lead to

$$a_2^2 = \frac{B_1^2D_1^2[B_1(1+2\beta)c_2 + D_1(1+3\alpha)b_2]}{2[\rho B_1^2D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2-B_1)D_1^2 - (1+\beta)^2(1+3\alpha)(D_2-D_1)B_1^2]},$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.19) when $|b_2| \leq 2$ and $|c_2| \leq 2$.

Multiplying (2.23) with $(3+5\beta)$ and (2.25) with $(1+2\alpha)$, and adding the results give

$$2a_3(2+7\alpha+3\beta+11\alpha\beta) = \frac{B_1}{2}(3+5\beta)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+\frac{c_1^2}{4}(3+5\beta)(B_2-B_1) + \frac{b_1^2}{4}(1+2\alpha)(D_2-D_1)$$

Substituting b_1 from (2.26) in the above equation give

$$2a_3(2+7\alpha+3\beta+11\alpha\beta) = \frac{B_1}{2}(3+5\beta)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+\frac{c_1^2}{4}\left((3+5\beta)(B_2-B_1) + \frac{(1+\beta)^2(D_2-D_1)B_1^2}{D_1^2(1+2\alpha)}\right)$$

which lead to

$$2\rho a_3 = \frac{1}{2}[B_1(3+5\beta)c_2 + D_1(1+2\alpha)b_2] \\ + \frac{c_1^2}{4}\left[(3+5\beta)(B_2-B_1) + \frac{(1+\beta)^2 B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],$$

where $\rho = 2+7\alpha+3\beta+11\alpha\beta$ and this yields the estimate given in (2.20). ∎

Theorem 2.4 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$, then*

$$|a_2| \leq \frac{B_1D_1\sqrt{2[B_1(3-2\beta)+D_1(1+3\alpha)]}}{\sqrt{|\rho B_1^2D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2|}} \tag{2.27}$$

and

$$|\rho a_3| \leq \frac{1}{2}B_1(\beta^2-11\beta+16) + D_1(1+2\alpha) + \frac{1}{2}(\beta^2-11\beta+16)|B_2-B_1| \\ + \frac{(2-\beta)^2B_1^2|D_2-D_1|}{D_1^2(1+2\alpha)} \tag{2.28}$$

where $\rho := 10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2$.

Proof. Let $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)), \tag{2.29}$$

Using

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots,$$

$$\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}$$

$$= 1 - (2-\beta)a_2 w + \Big((8(1-\beta) + \frac{1}{2}\beta(\beta+5))a_2^2 - 2(3-2\beta)a_3\Big)w^2 + \cdots,$$

and equations (2.10), (2.11) and (2.29) will yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1 c_1, \tag{2.30}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.31}$$

$$-(2-\beta)a_2 = \frac{1}{2}D_1 b_1 \tag{2.32}$$

and

$$[8(1-\beta) + \frac{\beta}{2}(\beta+5)]a_2^2 - 2(3-2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.33}$$

It follows from (2.30) and (2.32) that

$$b_1 = -\frac{B_1(2-\beta)}{D_1(1+2\alpha)}c_1. \tag{2.34}$$

Multiplying (2.31) with $(3-2\beta)$ and (2.33) with $(1+3\alpha)$, and adding the results give

$$a_2^2\left(5 - \frac{7\beta}{2} + 18\alpha - \frac{25\alpha\beta}{2} + \frac{\beta^2}{2} + \frac{3\alpha\beta^2}{2}\right) = \frac{B_1}{2}(3-2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$
$$+ \frac{c_1^2}{4}(3-2\beta)(B_2 - B_1) + \frac{b_1^2}{4}(1+3\alpha)(D_2 - D_1)$$

Substituting c_1 from (2.30) and b_1 from (2.34) in the above equation give

$$a_2^2\left(5 - \frac{7\beta}{2} + 18\alpha - \frac{25\alpha\beta}{2} + \frac{\beta^2}{2} + \frac{3\alpha\beta^2}{2}\right)$$
$$- \frac{a^2(1+2\alpha)^2}{B_1^2}\left((3-2\beta)(B_2 - B_1) + \frac{B_1^2(2-\beta)^2}{D_1^2(1+2\alpha)^2}(1+3\alpha)(D_2 - D_1)\right)$$

$$= \frac{B_1}{2}(3-2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$

which lead to

$$a_2^2 = \frac{B_1^2 D_1^2[B_1(3-2\beta)c_2 + D_1(1+3\alpha)b_2]}{\rho B_1^2 D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2},$$

which again in view of $|b_2| \leq 2$ and $|c_2| \leq 2$ gives the desired estimate on $|a_2|$ as asserted in (2.27). Multiplying (2.31) with $[8(1-\beta) + \frac{\beta}{2}(\beta+5)]$ and (2.33) with $(1+2\alpha)$, and adding the results give

$$a_3(10 + 36\alpha - 7\beta - 25\alpha\beta + \beta^2 + 3\alpha\beta^2) = \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+ \frac{c_1^2}{4}\left[8(1-\beta) + \frac{\beta}{2}(\beta+5)\right](B_2 - B_1) + \frac{b_1^2}{4}(1+2\alpha)(D_2 - D_1)$$

Substituting b_1 from (2.34) in the above equation give

$$a_3(10 + 36\alpha - 7\beta - 25\alpha\beta + \beta^2 + 3\alpha\beta^2) = \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+ \frac{c_1^2}{4}\left[8(1-\beta) + \frac{\beta}{2}(\beta+5)\right](B_2 - B_1) + \frac{c_1^2(2-\beta)^2 B_1^2(D_2 - D_1)}{4D_1^2(1+2\alpha)}$$

which lead to

$$\rho a_3 = \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+ \frac{c_1^2}{4}\left[\frac{1}{2}(\beta^2 - 11\beta + 16)(B_2 - B_1) + \frac{(2-\beta)^2 B_1^2(D_2 - D_1)}{D_1^2(1+2\alpha)}\right]$$

where $\rho := 10 + 36\alpha - 7\beta - 25\alpha\beta + \beta^2 + 3\alpha\beta^2$ and this yields the estimate given in (2.28). ∎