UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2012/2013 Academic Session

June 2013

# MSG 284 – Introduction to Geometric Modelling
# *[Pengenalan kepada Pemodelan Geometri]*

Duration : 3 hours
*[Masa : 3 jam]*

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Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions**: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]*

1. (a) Find a polynomial function that interpolates the points $(1, 3)$, $(2, 2)$ and $(4, 2)$.

(b) Let $\boldsymbol{P}(u)$ be a parametric spline curve defined as

$$\boldsymbol{P}(u) = \begin{cases} \boldsymbol{F}(u), & u \leq a \\ \boldsymbol{G}(u), & u > a \end{cases}$$

where $a \in \mathbb{R}$. Define the geometric continuities $G^0$, $G^1$, $G^2$ of curve $\boldsymbol{P}(u)$ at $u = a$.

(c) Let $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, $\kappa$ and $\tau$ be the unit tangent, principal normal, binormal, curvature and torsion of a regular curve $\boldsymbol{P}(s)$ at arc length parameter $s \geq 0$. Show that the derivative

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}s} = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}.$$

[100 marks]

*1. (a) Cari fungsi polinomial yang menginterpolasi titik-titik* $(1, 3)$, $(2, 2)$ *dan* $(4, 2)$.

*(b) Katakan* $\boldsymbol{P}(u)$ *ialah satu lengkung splin berparameter ditakrif sebagai*

$$\boldsymbol{P}(u) = \begin{cases} \boldsymbol{F}(u), & u \leq a \\ \boldsymbol{G}(u), & u > a \end{cases}$$

*di mana* $a \in \mathbb{R}$. *Takrifkan keselanjaran geometri* $G^0$, $G^1$, $G^2$ *bagi lengkung* $\boldsymbol{P}(u)$ *pada* $u = a$.

*(c) Katakan* $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, $\kappa$ *dan* $\tau$ *ialah tangent unit, normal prinsipal, binormal, kelengkungan dan kilasan bagi suatu lengkung nalar* $\boldsymbol{P}(s)$ *pada parameter panjang lengkok* $s \geq 0$. *Tunjukkan bahawa terbitan*

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}s} = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}.$$

*[100 markah]*

2. (a) Consider a plane Bézier polynomial of degree $n$

$$\boldsymbol{P}(t) = \sum_{i=0}^{n} \boldsymbol{C}_i B_i^n(t), \qquad 0 \le t \le 1,$$

where $\boldsymbol{C}_i \in \mathbb{R}^2$ are control points and $B_i^n(t)$ denote the Bernstein polynomial

$$B_i^n(t) = \frac{n!}{i!\,(n-i)!} t^i (1-t)^{n-i}, \qquad \text{for} \quad i = 0, 1, \ldots, n.$$

State the reason of why the Bézier curve lies entirely within the convex hull of its control polygon.

(b) Given a cubic Bézier curve

$$\boldsymbol{P}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_0^3(t) + \begin{pmatrix} 2 \\ \alpha \end{pmatrix} B_1^3(t) + \begin{pmatrix} \beta \\ 4 \end{pmatrix} B_2^3(t) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} B_3^3(t), \qquad 0 \le t \le 1,$$

where $\alpha$, $\beta \in \mathbb{R}$ and $B_i^3(t)$, $i = 0, 1, 2, 3$, are the Bernstein polynomials of degree $3$. Calculate the $\alpha$ and $\beta$ such that the curve $\boldsymbol{P}(t)$ can be reduced to a quadratic curve.

(c) Given a piecewise curve

$$\boldsymbol{P}(u) = \begin{cases} \boldsymbol{F}(u), & 0 \le u \le 1 \\ \boldsymbol{G}(u), & 1 < u \le 3. \end{cases}$$

The curve segments $\boldsymbol{F}(u)$ and $\boldsymbol{G}(u)$ can be represented in terms of a local parameter $t \in [0, 1]$ as

$$\boldsymbol{F}(t) = \boldsymbol{C}_0(1-t)^2 + \boldsymbol{C}_1 2t(1-t) + \boldsymbol{C}_2 t^2,$$

$$\boldsymbol{G}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}(1 - 3t^2 + 2t^3) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(t - 2t^2 + t^3) + \begin{pmatrix} 3 \\ 1 \end{pmatrix}(t^3 - t^2) + \begin{pmatrix} 3 \\ 2 \end{pmatrix}(3t^2 - 2t^3),$$

where $\boldsymbol{C}_i \in \mathbb{R}^2$, $i = 0, 1, 2$. Determine all the $\boldsymbol{C}_i$ such that $\boldsymbol{P}(u)$ is a $C^1$ continuous curve with the derivative

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0) = (1,\ 0).$$

[100 marks]

2. (a) *Pertimbangkan satu polinomial Bézier satah berdarjah* $n$

$$\boldsymbol{P}(t) = \sum_{i=0}^{n} \boldsymbol{C}_i B_i^n(t), \qquad 0 \le t \le 1,$$

*di mana* $\boldsymbol{C}_i \in \mathbb{R}^2$ *adalah titik kawalan dan* $B_i^n(t)$ *menandakan polinomial Bernstein*

$$B_i^n(t) = \frac{n!}{i!\,(n-i)!} t^i \left(1-t\right)^{n-i}, \qquad bagi \quad i = 0, 1, \ldots, n.$$

*Nyatakan sebab mengapa lengkung Bézier terletak sepenuhnya dalam hul cembung poligon kawalan.*

(b) *Diberi suatu lengkung Bézier kubik*

$$\boldsymbol{P}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_0^3(t) + \begin{pmatrix} 2 \\ \alpha \end{pmatrix} B_1^3(t) + \begin{pmatrix} \beta \\ 4 \end{pmatrix} B_2^3(t) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} B_3^3(t), \qquad 0 \le t \le 1,$$

*di mana* $\alpha$, $\beta \in \mathbb{R}$ *dan* $B_i^3(t)$, $i = 0, 1, 2, 3$, *ialah polinomial Bernstein berdarjah* 3. *Kirakan* $\alpha$ *dan* $\beta$ *supaya lengkung* $\boldsymbol{P}(t)$ *boleh dikurangkan kepada lengkung kuadratik.*

(c) *Diberi satu lengkung bercebisan*

$$\boldsymbol{P}(u) = \begin{cases} \boldsymbol{F}(u), & 0 \le u \le 1 \\ \boldsymbol{G}(u), & 1 < u \le 3. \end{cases}$$

*Lengkung segmen* $\boldsymbol{F}(u)$ *dan* $\boldsymbol{G}(u)$ *boleh diwakili dalam sebutan parameter setempat* $t \in [0, 1]$ *sebagai*

$$\boldsymbol{F}(t) = \boldsymbol{C}_0(1-t)^2 + \boldsymbol{C}_1 2t(1-t) + \boldsymbol{C}_2 t^2,$$

$$\boldsymbol{G}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}(1-3t^2+2t^3) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(t-2t^2+t^3) + \begin{pmatrix} 3 \\ 1 \end{pmatrix}(t^3-t^2) + \begin{pmatrix} 3 \\ 2 \end{pmatrix}(3t^2-2t^3),$$

*di mana* $\boldsymbol{C}_i \in \mathbb{R}^2$, $i = 0, 1, 2$. *Tentukan semua* $\boldsymbol{C}_i$ *supaya* $\boldsymbol{P}(u)$ *ialah satu lengkung berkeselanjaran* $C^1$ *dengan terbitan*

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0) = (1, 0).$$

*[100 markah]*

3. Let $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n+k})$ be a non-decreasing knot vector where $n$ and $k$ are positive integers. The normalized B-spline basis functions of order $k$ are defined recursively by

$$N_i^1(u) = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad \text{for} \quad k > 1,$$

where $i = 0, 1, \ldots, n$.

(a) Suppose $\boldsymbol{u} = (0, 3, 6, 9)$. Sketch the function $N_0^3(u)$ and calculate the function values at the given knots.

(b) Consider a B-spline curve of order 3 on Cartesian plane with knot vector $\boldsymbol{u} = (-2, -1, 0, 1, 1, 2)$

$$\boldsymbol{P}(u) = \boldsymbol{D}_0 N_0^3(u) + \boldsymbol{D}_1 N_1^3(u) + \boldsymbol{D}_2 N_2^3(u), \qquad 0 \le u \le 1.$$

(i) Show that $N_0^3(u) + N_1^3(u) + N_2^3(u) = 1$, for $0 \le u \le 1$.

(ii) Find the coefficients $\boldsymbol{D}_i \in \mathbb{R}^2$, $i = 0, 1, 2$, such that the coordinate $y$ of $\boldsymbol{P}(u)$ is non-negative with

$$\boldsymbol{P}(0) = (1, 0), \qquad \boldsymbol{P}(1) = (4, 1),$$

and the dot product of the first derivatives

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0) \bullet \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1) = 0$$

where the length of vector $\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0)$ is twice the length of vector $\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1)$.

[100 marks]

3. *Katakan* $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n+k})$ *ialah suatu vektor simpulan tak menyusut di mana* $n$ *dan* $k$ *adalah integer positif. Fungsi asas splin-B ternormal berperingkat* $k$ *ditakrif secara rekursi oleh*

$$N_i^1(u) = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \text{di tempat lain} \end{cases}$$

*dan*

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad \text{bagi} \quad k > 1,$$

*di mana* $i = 0, 1, \ldots, n$.

(a) *Andaikan* $\boldsymbol{u} = (0, 3, 6, 9)$. *Lakarkan fungsi* $N_0^3(u)$ *dan kirakan nilai fungsi pada simpulan yang diberikan.*

(b) *Pertimbangkan suatu lengkung splin-B berperingkat* 3 *pada satah Cartesan dengan vektor simpulan* $\boldsymbol{u} = (-2, -1, 0, 1, 1, 2)$

$$\boldsymbol{P}(u) = \boldsymbol{D}_0 N_0^3(u) + \boldsymbol{D}_1 N_1^3(u) + \boldsymbol{D}_2 N_2^3(u), \qquad 0 \le u \le 1.$$

(i) *Tunjukkan bahawa* $N_0^3(u) + N_1^3(u) + N_2^3(u) = 1$, *untuk* $0 \le u \le 1$.

(ii) *Cari koefisien* $\boldsymbol{D}_i \in \mathbb{R}^2$, $i = 0, 1, 2$, *supaya koordinat* $y$ *bagi* $\boldsymbol{P}(u)$ *adalah tak negatif dengan*

$$\boldsymbol{P}(0) = (1, 0), \qquad \boldsymbol{P}(1) = (4, 1),$$

*dan hasil darab bintik bagi terbitan pertama*

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0) \bullet \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1) = 0$$

*di mana panjang vektor* $\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(0)$ *adalah dua kali panjang vektor* $\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1)$.

*[100 markah]*

4. (a) Consider a biquadratic Bézier surface on Cartesian coordinate space

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \boldsymbol{C}_{i,j} \, B_j^2(v) \, B_i^2(u), \qquad 0 \le u, v \le 1,$$

where $B_s^2(t)$, $0 \le t \le 1$, $s = 0, 1, 2$, indicate the Bernstein polynomials of degree $2$ and $\boldsymbol{C}_{i,j}$ are the control points given as

$\boldsymbol{C}_{0,0} = (1, 1, 1)$, $\boldsymbol{C}_{0,1} = (1, 2, 2)$, $\boldsymbol{C}_{0,2} = (1, 3, 1)$,

$\boldsymbol{C}_{1,0} = (2, 1, 2)$, $\boldsymbol{C}_{1,1} = (2, 2, 4)$, $\boldsymbol{C}_{1,2} = (2, 3, 2)$,

$\boldsymbol{C}_{2,0} = (4, 1, 1)$, $\boldsymbol{C}_{2,1} = (3, 2, 2)$, $\boldsymbol{C}_{2,2} = (4, 3, 1)$.

(i) Find the surface point $\boldsymbol{S}(u, v)$ that has maximum value of coordinate $z$.

(ii) Express the curve on surface $\boldsymbol{S}$ along a parametric line $u = v$ in Bézier representation.

(b) Consider a bilinearly blended Coons patch $\boldsymbol{F}(u, v)$, $0 \le u, v \le 1$, which

$\boldsymbol{F}(0, 0) = (1, 1, 3)$, $\boldsymbol{F}(0, 1) = (1, 5, 1)$,

$\boldsymbol{F}(1, 0) = (5, 1, 1)$, $\boldsymbol{F}(1, 1) = (5, 5, 1)$.

Suppose the boundaries of patch $\boldsymbol{F}$ at $u = 0$ and $v = 0$ are Bézier quadratics

$$\boldsymbol{F}(0, v) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} (1-v)^2 + \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} 2v(1-v) + \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} v^2,$$

$$\boldsymbol{F}(u, 0) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} (1-u)^2 + \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} 2u(1-u) + \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} u^2,$$

while the other two boundaries are linear polynomials. Evaluate the unit normal vector to the patch $\boldsymbol{F}$ at $(u, v) = (0.5, 0.5)$.

[100 marks]

4. (a) *Pertimbangkan suatu permukaan Bézier dwikuadratik pada ruang koordinat Cartesian*

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{C}_{i,j}\,B_j^2(v)\,B_i^2(u)\,,\qquad 0\le u,v\le 1\,,$$

*di mana* $B_s^2(t)$, $0\le t\le 1$, $s=0,\ 1,\ 2$, *menandakan polinomial Bernstein berdarjah* $2$ *dan* $\boldsymbol{C}_{i,j}$ *adalah titik-titik kawalan yang diberikan sebagai*

$\boldsymbol{C}_{0,0}=(1,\ 1,\ 1)$, $\boldsymbol{C}_{0,1}=(1,\ 2,\ 2)$, $\boldsymbol{C}_{0,2}=(1,\ 3,\ 1)$,

$\boldsymbol{C}_{1,0}=(2,\ 1,\ 2)$, $\boldsymbol{C}_{1,1}=(2,\ 2,\ 4)$, $\boldsymbol{C}_{1,2}=(2,\ 3,\ 2)$,

$\boldsymbol{C}_{2,0}=(4,\ 1,\ 1)$, $\boldsymbol{C}_{2,1}=(3,\ 2,\ 2)$, $\boldsymbol{C}_{2,2}=(4,\ 3,\ 1)$.

(i) *Cari titik permukaan* $\boldsymbol{S}(u,v)$ *yang mempunyai nilai maksimum koordinat* $z$.

(ii) *Nyatakan lengkung pada permukaan* $\boldsymbol{S}$ *di sepanjang garis parameter* $u=v$ *dalam perwakilan Bézier.*

(b) *Pertimbangkan satu tampalan Coons teraduan dwilinear* $\boldsymbol{F}(u,\ v)$, $0\le u,v\le 1$, *di mana*

$\boldsymbol{F}(0,\ 0)=(1,\ 1,\ 3)$, $\boldsymbol{F}(0,\ 1)=(1,\ 5,\ 1)$,

$\boldsymbol{F}(1,\ 0)=(5,\ 1,\ 1)$, $\boldsymbol{F}(1,\ 1)=(5,\ 5,\ 1)$.

*Andaikan sempadan tampalan* $\boldsymbol{F}$ *pada* $u=0$ *dan* $v=0$ *ialah kuadratik Bézier*

$$\boldsymbol{F}(0,\ v)=\begin{pmatrix}1\\1\\3\end{pmatrix}(1-v)^2+\begin{pmatrix}1\\3\\3\end{pmatrix}2v(1-v)+\begin{pmatrix}1\\5\\1\end{pmatrix}v^2\,,$$

$$\boldsymbol{F}(u,\ 0)=\begin{pmatrix}1\\1\\3\end{pmatrix}(1-u)^2+\begin{pmatrix}3\\1\\3\end{pmatrix}2u(1-u)+\begin{pmatrix}5\\1\\1\end{pmatrix}u^2\,,$$

*manakala dua sempadan lain adalah polinomial linear. Nilaikan vektor unit normal kepada tampalan* $\boldsymbol{F}$ *pada* $(u,\ v)=(0.5,\ 0.5)$.

*[100 markah]*

**- ooo O ooo -**