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UNIVERSITI SAINS MALAYSIA

First Semester Examination
2012/2013 Academic Session

January 2013

MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini].

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Let Y be a continuous random variable with $f_Y(y) = ye^{-y}, 0 \le y$. Show that the moment generating function $M_Y(t) = \frac{1}{(1-t)^2}$.
Hence, find the mean and standard deviation of the random variable Y.

[35 marks]

(b) Let X and Y have joint probability density function
$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$
Suppose $A = X/Y$ and $B = Y$. Find the joint probability density function of (A, B) and hence the marginal probability density of A and B.

[35 marks]

(c) Let X have the probability density function $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere. Find the cumulative distribution function and the probability density function of $Y = -2\ln X^4$.

[30 marks]

1. (a) Andaikan Y suatu pembolehubah rawak selanjar dengan $f_Y(y) = ye^{-y}, 0 \le y$. *Tunjukkan bahawa fungsi penjana momen* $M_Y(t) = \frac{1}{(1-t)^2}$.
Seterusnya, cari min dan sisihan piawai bagi pembolehubah rawak Y.

[35 markah]

(b) Biarkan X dan Y mempunyai fungsi ketumpatan kebarangkalian tercantum
$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & \textit{selainnya} \end{cases}$$
Andaikan A = X/Y dan B = Y. Cari fungsi ketumpatan kebarangkalian tercantum (A, B) dan seterusnya fungsi ketumpatan kebarangkalian sut bagi A dan B.

[35 markah]

(c) Biarkan X mempunyai fungsi ketumpatan kebarangkalian $f(x) = 4x^3$, $0 < x < 1$, *sifar selainnya. Cari fungsi taburan longgokan dan fungsi ketumpatan kebarangkalian bagi* $Y = -2\ln X^4$.

[30 markah]

2. (a) (i) If the random variable X has a standard normal distribution, show using the distribution function technique that $Y = X^2$ has a chi-square distribution with degree of freedom one.

(ii) If $Y_1, Y_2, \ldots, Y_n$ are random variables from chi-square distribution with degree of freedom one, find the distribution of $Z = \sum_{i=1}^{n} Y_i$ using the moment generating function approach.

[40 marks]

(b) A random sample of size $n = 6$ is taken from the pdf $f_Y(y) = 3y^2, 0 \le y \le 1.$ Find the probability of the fifth order statistics $P(Y_5^{'} > 0.75).$

[30 marks]

(c) Let $Y_1, Y_2, ..., Y_n$ be a random sample from the exponential pdf $f_Y(y) = e^{-y}, y > 0.$ What is the smallest n for which $P(Y_{\min} < 0.2) > 0.9$?

[30 marks]

2. (a) (i) Jika pembolehubah rawak X mempunyai taburan normal piawai, tunjukkan dengan menggunakan teknik fungsi taburan bahawa $Y = X^2$ mempunyai suatu taburan chi-kuasadua dengan darjah kebebasan satu.

(ii) Jika $Y_1, Y_2, \ldots, Y_n$ adalah pembolehubah rawak dari taburan khi-kuasadua dengan darjah kebebasan satu, cari taburan bagi $Z = \sum_{i=1}^{n} Y_i$ dengan menggunakan kaedah fungsi penjana momen.

[40 markah]

(b) Suatu sampel rawak bersaiz n = 6 diambil dari fungsi ketumpatan kebarangkalian $f_Y(y) = 3y^2, 0 \le y \le 1.$ Cari kebarangkalian bagi statistik tertib kelima. $P(Y_5^{'} > 0.75).$

[30 markah]

(c) Biarkan $Y_1, Y_2, ..., Y_n$ sebagai suatu sampel rawak dari taburan eksponen dengan fungsi ketumpatan kebarangkalian $f_Y(y) = e^{-y}, y > 0.$ Apakah nilai terkecil n dengan keadaan $P(Y_{\min} < 0.2) > 0.9$?

[30 markah]

3. (a) Let W_n be a random variable with mean μ and variance b/n^p, where $p > 0$, μ and b are constants (not functions of n). Prove that W_n converges in probability to μ. [Hint: Use Chebyshev's inequality]

[20 marks]

(b) Let $\bar{X}$ be the mean of n independent random variables $X_1, X_2, \ldots, X_n$ from the exponential distribution with parameter θ.

(i) Show that $\bar{X}$ is an unbiased point estimator of $1/\theta$.

(ii) Using the moment generating function technique, determine the distribution of $\bar{X}$.

(iii) Use part (ii) to show that $Y = 2n\theta\bar{X}$ has a χ^2 distribution with $2n$ degrees of freedom.

[40 marks]

(c) Let $X_1 = 0.23$, $X_2 = 0.65$, $X_3 = 0.10$ and $X_4 = 0.42$ be a random sample of size four from the probability density function

$$f_X(x;\theta) = \theta x^{\theta-1},\ 0 \le x \le 1.$$

Find the method of moments estimate for θ.

[40 marks]

3. (a) Biarkan W_n sebagai suatu pembolehubah rawak dengan min μ dan varians b/n^p, yang mana $p > 0$, μ and b adalah pemalar (bukan fungsi n). Buktikan bahawa W_n menumpu secara kebarangkalian kepada μ. [Petunjuk: Guna ketaksamaan Chebyshev]

[20 markah]

(b) Biarkan $\bar{X}$ sebagai min bagi n pembolehubah rawak tak bersandar $X_1, X_2, \ldots, X_n$ dari suatu taburan eksponen dengan parameter θ.

(i) Tunjukkan bahawa $\bar{X}$ adalah suatu penganggar titik saksama bagi $1/\theta$.

(ii) Guna teknik fungsi penjana momen bagi menentukan taburan bagi $\bar{X}$.

(iii) Guna bahagian (ii) bagi menunjukkan bahawa $Y = 2n\theta\bar{X}$ mempunyai taburan χ^2 dengan darjah kebebasan $2n$.

[40 markah]

(c) Biarkan $X_1 = 0.23$, $X_2 = 0.65$, $X_3 = 0.10$ dan $X_4 = 0.42$ adalah suatu sampel rawak bersaiz empat dari fungsi ketumpatan kebarangkalian

$$f_X(x;\theta) = \theta x^{\theta-1},\ 0 \le x \le 1.$$

Cari anggaran bagi θ dengan menggunakan kaedah momen.

[40 markah]

4. (a) Let X be a random variable from the Poisson distribution with parameter θ and assume that the prior distribution of the parameter Θ has a gamma distribution with parameters α and λ which are known. Find the posterior distribution of Θ given $X = x$ and hence write the mean of the posterior distribution of Θ.

[40 marks]

(b) Suppose the random variables $Y_1, Y_2, \ldots, Y_n$ denote the number of successes or failures (1 or 0) in each of n independent trials, where p = P(success occurs at given trial) is an unknown parameter such that $p_{Y_i}(k;p) = p^k(1-p)^{1-k}, \quad k=0,1; \quad 0 < p < 1$

If $Y = Y_1 + Y_2 + \ldots + Y_n$ = total number of successes and define $\hat{p} = \frac{Y}{n}$.

(i) Show that $\hat{p}$ is an unbiased estimator for p.

(ii) Calculate the variance for $\hat{p}$.

(iii) Compute the Cramer-Rao lower bound for $p_{Y_i}(k;p)$

(iv) Compare the results in (ii) and (iii) and discuss.

[40 marks]

(c) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a probability distribution with density function $f_X(x;\theta) = \theta x^{\theta-1}, \; 0 \le x \le 1$. If a and b are two number such that $0 < a < b < 1$, construct a $100\gamma\%$ confidence interval for θ using the pivotal quantity method.

[20 marks]

4. (a) Biarkan X sebagai suatu pembolehubah rawak dari taburan Poisson dengan parameter θ dan andaikan taburan prior bagi parameter Θ mempunyai taburan gama dengan parameter α dan λ yang diketahui. Cari taburan posterior bagi Θ diberi X = x dan seterusnya tulis min bagi taburan posterior untuk Θ.

[40 markah]

(b) Andaikan pembolehubah rawak $Y_1, Y_2, \ldots, Y_n$ adalah bilangan kejayaan atau kegagalan (1 or 0) dalam setiap n percubaan yang tak bersandar, yang mana p = P(kejayaan berlaku pada suatu percubaan yang diberi) adalah suatu parameter yang tidak diketahui supaya $p_{Y_i}(k;p) = p^k(1-p)^{1-k}, \quad k=0,1; \quad 0 < p < 1$

Jika $Y = Y_1 + Y_2 + \ldots + Y_n$ =jumlah bilangan kejayaan dan takrifkan $\hat{p} = \frac{Y}{n}$.

(i) Tunjukkan bahawa $\hat{p}$ suatu penganggar saksama bagi p.

(ii) Kira varians bagi $\hat{p}$.

(iii) Kira batas bawah Cramer-Rao bagi $p_{Y_i}(k;p)$

(iv) Bandingkan keputusan dalam (ii) dan (iii) dan bincang.

[40 markah]

(c) *Biarkan* $X_1, X_2, \ldots, X_n$ *suatu sampel rawak bersaiz n dari taburan kebarangkalian dengan fungsi ketumpatan* $f_X(x;\theta) = \theta x^{\theta-1},\ 0 \le x \le 1.$ *Jika a dan b adalah dua nombor supaya* $0 < a < b < 1$*, bina suatu selang keyakinan 100*γ *% bagi* θ *dengan menggunakan kaedah kuantiti pangsian.*

[20 markah]

5. (a) Suppose n = 36 observations are taken from a normal distribution where $\sigma = 8.0$
for the purpose of testing $H_0 : \mu = 60$ versus $H_1 : \mu \neq 60$ at the $\alpha = 0.07$ level of significance. The lead investigator skipped statistics class the day decision rules were being discussed and intends to reject H_0 if $\bar{y}$ falls in the interval $(60 - \bar{y}^*, 60 + \bar{y}^*)$.

(i) Find $\bar{y}^*$.

(ii) What is the power of the test when $\mu = 62$?

[40 marks]

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution $P(\mu)$.

(i) Find the critical region of the most powerful test of size α for testing $H_0 : \mu = 1.5$ against $H_1 : \mu = 2.$

(ii) If $\alpha = 0.05$ and $n = 10$ for $H_0 : \mu = 1$ against $H_1 : \mu = 2$, determine the most powerful test.

[40 marks]

(c) Let $X_1, X_2, \ldots X_9$ be a random sample of size 9 from a distribution that is $N(\mu, \sigma^2)$. If σ is known, find the length of a 95 percent confidence interval for μ if this interval is based on the random variable $\sqrt{9}(\bar{X} - \mu)/\sigma$.

[20 marks]

5. (a) *Andaikan n = 36 cerapan diambil dari suatu taburan normal yang mana σ = 8.0 bagi tujuan menguji* $H_0 : \mu = 60$ lawan $H_1 : \mu \neq 60$ *pada aras keertian α = 0.07. Penyelidik utama mengelak kelas pada hari peraturan keputusan dibincang dan ingin menolak* H_0 jika $\bar{y}$ *terletak dalam selang* $(60 - \bar{y}^*, 60 + \bar{y}^*)$.

(i) *Cari* $\bar{y}^*$.

(ii) *Apakah kuasa ujian apabila μ = 62?*

[40 markah]

(b) *Biarkan* $X_1, X_2, \ldots, X_n$ *suatu sampel rawak bersaiz n dari taburan Poisson* $P(\mu)$.

(i) *Cari rantau genting bagi ujian paling berkuasa bersaiz α bagi menguji* $H_0 : \mu = 1.5$ lawan $H_1 : \mu = 2$.

(ii) *Jika α = 0.05 dan n = 10 bagi* $H_0 : \mu = 1$ lawan $H_1 : \mu = 2$, *tentukan ujian paling berkuasa.*

[40 markah]

(c) *Biarkan* $X_1, X_2, \ldots X_9$ *sebagai suatu sampel rawak bersaiz 9 dari suatu taburan* $N(\mu, \sigma^2)$. *Jika σ diketahui, cari panjang bagi selang keyakinan 95 peratus bagi μ jika selang ini adalah berdasarkan pembolehubah rawak* $\sqrt{9}(\bar{X} - \mu)/\sigma$.

[20 markah]

LAMPIRAN

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\dots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\,e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\dots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\dots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\dots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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