UNIVERSITI SAINS MALAYSIA

First Semester Examination
2012/2013 Academic Session

January 2013

MAT 222 - Differential Equations II
[Persamaan Pembezaan II]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Given a system of first order linear differential equations

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

find the general solution.

(b) Using the variation of parameters, find the particular solution of

$$\frac{dx}{dt} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} x + e^{2t} \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

(c) Given two vectors

$$x^{1} \quad t = \begin{pmatrix} 1 \\ t \end{pmatrix} \quad \text{and} \quad x^{2} \quad t = e^{t} \begin{pmatrix} 1 \\ t \end{pmatrix},$$

show that the two vectors are linearly dependent at each point in the interval $0 \le t \le 1$. However, show that the two vectors are linearly independent on $0 \le t \le 1$.

[100 marks]

1. (a) Di beri satu sistem persamaan pembezaan linear peringkat satu

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

dapatkan penyelesaian am.

(b) Mengguna ubahan parameter, dapatkan penyelesaian khusus untuk

$$\frac{dx}{dt} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} x + e^{2t} \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

(c) Di beri dua vektor

$$x^{1} \quad t = \begin{pmatrix} 1 \\ t \end{pmatrix} \quad \textit{dan} \quad x^{2} \quad t = e^{t} \begin{pmatrix} 1 \\ t \end{pmatrix},$$

tunjukkan kedua-dua vektor bersandar linear pada setiap titik dalam selang $0 \le t \le 1$. *Sebaliknya, tunjukkan bahawa kedua-dua vektor tak bersandar linear atas* $0 \le t \le 1$.

[100 markah]

2. (a) Given a system of first order autonomous system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y) \qquad (*)$$

Explain the difference between the solution to (*) and the trajectories to (*)

(b) Let (x_0, y_0) be the isolated equilibrium point of (*)

(i) Give a definition of stability of (x_0, y_0)

(ii) Give a definition of asymptotic stability of (x_0, y_0)

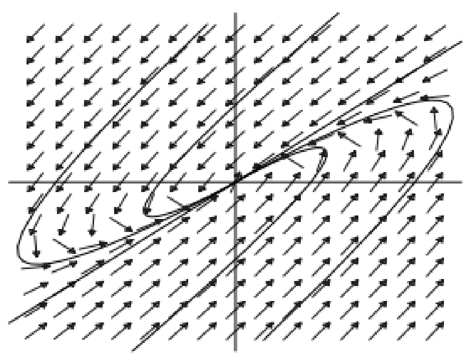
(c) Consider the system

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -x - y$$

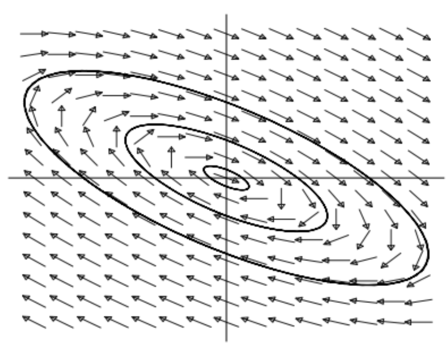
(i) Find the fundamental solutions
(ii) Sketch the trajectories on the phase plane
(iii) Discuss the stability of the equilibrium point.

(d) Given the following phase portraits, classify the equilibrium point $(0, 0)$

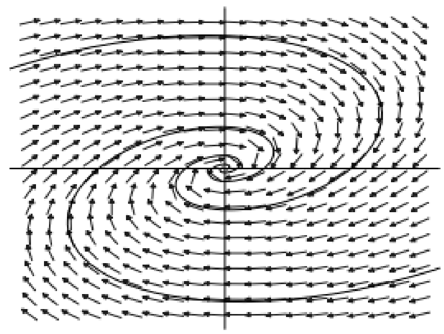
(i)



(ii)



(iii)



[100 marks]

2\. (a) *Di beri satu system otonomi peringkat satu*

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y) \qquad (*)$$

Terangkan perbezaan antara penyelesaian kepada (*) dan trajektori kepada (*)

(b) *Andaikan* (x_0, y_0) *titik seimbang terpencil kepada (*)*

(i) *Beri satu takrif kestabilan untuk* (x_0, y_0)

(ii) *Beri satu takrif kestabilan asimptot untuk* (x_0, y_0)

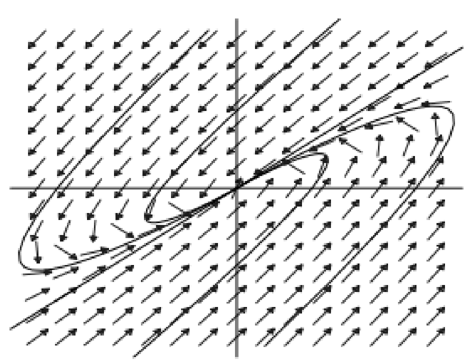
(c) *Pertimbangan sistem*

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -x - y$$

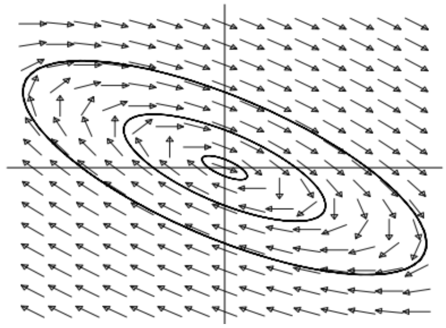
(i) *Dapatkan penyelesaian-penyelesaian asas*
(ii) *Lakarkan trajektori atas satah fasa*
(iii) *Bincangkan kestabilan titik seimbang.*

(d) *Di beri potrat-potrat fasa berikut, kelaskan titik seimbang* $(0, 0)$

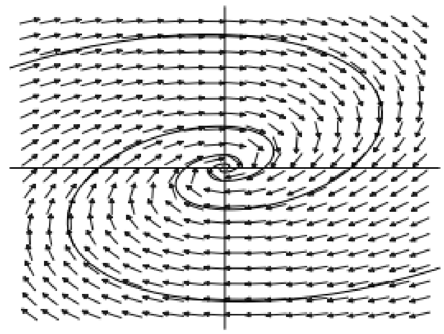
(i)



(iii)



(iii)



[100 markah]

3. The regular Sturm-Liouville problem connected with the Euler operator can be written as

$$\frac{d^2}{dx^2} y(x) + \lambda y(x) = 0, \quad 0 < x < 1$$

together with Neumann condition on the left and Dirichlet condition on the right

$$y_x(0) = 0, \quad y(1) = 0$$

Evaluate the generalized Fourier series expansion in terms of the orthonormal eigenfunctions from the above problem for the function

$f(x) = x$

[100 marks]

3. *Masalah Sturm-Liouville nalar yang berkaitan dengan operator Euler boleh ditulis seperti*

$$\frac{d^2}{dx^2} y(x) + \lambda y(x) = 0, \quad 0 < x < 1$$

bersama dengan syarat Neumann di sebelah kiri dan syarat Dirichlet di sebelah kanan

$$y_x(0) = 0, \quad y(1) = 0$$

Nilaikan siri am Fourier untuk fungsi

$f(x) = x$

[100 markah]

4. (a) Given the equations $yu_{xx} + u_{yy} = 0$ what are the conditions on y that will classed the above equation into hyperbolic, parabolic and elliptic types?

(b) Put the following equation $y^2u_{xx} - x^2u_{yy} = 0, \quad x \in (0,\infty), \quad y \in (0,\infty)$ into the canonical form $u_{\zeta\eta} = f(\zeta,\eta,u,u_\zeta,u_\eta)$ by using new coordinates $\zeta(x,y) = x^2 + y^2 =$ constant, $\eta(x,y) = y^2 - x^2 =$ constant

(c) The wave equation can be written as $u_{tt} = c^2u_{xx}$ together with the initial-boundary conditions

$$u(0,t) = u(L,t) = 0 \text{ for all } t$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x), \quad 0 < x < L$$

By a change of variables, $\zeta = x - ct, \eta = x + ct$, find the d'Alembert solution to the equation.

[100 marks]

4. (a) Di beri persamaan $yu_{xx} + u_{yy} = 0$ *apakah syarat ke atas* y *yang akan mengkelaskan persamaan di atas kepada jenis-jenis hiperbolik, parabolik dan eliptik?*

(b) Letakkan persamaan berikut $y^2u_{xx} - x^2u_{yy} = 0, \quad x \in (0,\infty), \quad y \in (0,\infty)$ *dalam bentuk kanonik* $u_{\zeta\eta} = f(\zeta,\eta,u,u_\zeta,u_\eta)$ *dengan menggunakan koordinat baru* $\zeta(x,y) = x^2 + y^2 =$ *konstant,* $\eta(x,y) = y^2 - x^2 =$ *konstant*

(c) Persamaan gelombang boleh ditulis seperti $u_{tt} = c^2u_{xx}$ *bersama dengan syarat-syarat sempadan*

$$u(0,t) = u(L,t) = 0 \text{ untuk semua } t$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x), \quad 0 < x < L$$

Dapatkan penyelesaian d'Alembert kepada persamaan gelombang.

[100 markah]

5. The temperature distribution $u(x,t)$ of a thin insulated rod can be modeled by

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1$$

The equation is subjected to the initial condition

$$u(x,0) = x[1-x]$$

and boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = 0$$

Evaluate the eigenvalued and corresponding orthonormalised eigenfunctions and write the general solution.

[100 marks]

5. Taburan suhu $u(x,t)$ *satu rod nipis berbalut boleh di model melalui*

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1$$

Persamaan ini tertakluk kepada syarat awal

$$u(x,0) = x[1-x]$$

dan syarat-syarat sempadan

$$u_x(0,t) = 0, \quad u(1,t) = 0$$

Nilaikan nilai-nilai eigen dan fungsi-fungsi eigen berortonomal yang sempadan dan tuliskan penyelesaiam am.

[100 markah]

- ooo O ooo -