UNIVERSITI SAINS MALAYSIA

First Semester Examination
2012/2013 Academic Session

January 2013

MAT 517 – Computational Linear Algebra
[Aljabar Linear Pengkomputeran]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Describe the QR method for solving the following linear algebra problems:

(i) Solve the linear system of equation $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $n \times n$ non-singular matrix and $\mathbf{b}$ is a vector in R^n.

(ii) Find the least squares solution of $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a full rank, $m \times n$ matrix with $m > n$, and $\mathbf{b}$ is a vector in R^m.

(iii) Compute approximation to the eigenvalues of a symmetric tridiagonal matrix $\mathbf{A}$ of the form

$$\mathbf{A} = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & b_3 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}.$$

[60 marks]

(b) Create zeros in the last 2 entries of the vector

$$\begin{pmatrix} 4 \\ -3 \\ -2 \\ -1 \\ -2 \end{pmatrix},$$

using

(i) Householder reflectors;

(ii) Givens rotators.

[40 marks]

1. *(a) Huraikan kaedah QR untuk menyelesaikan masalah-masalah aljabar linear berikut:*

(i) Selesaikan sistem persamaan linear $\mathbf{Ax} = \mathbf{b}$ *di mana* $\mathbf{A}$ *ialah suatu matriks tak singular* $n \times n$ *dan* $\mathbf{b}$ *ialah vektor dalam* R^n.

(ii) Cari penyelesaian kuasa dua terkecil bagi $\mathbf{Ax} = \mathbf{b}$ *di mana* $\mathbf{A}$ *ialah matriks pangkat penuh* $m \times n$ *dengan* $m > n$*, dan* $\mathbf{b}$ *ialah vektor dalam* R^m.

(iii) Kira penghampiran kepada nilai-nilai eigen matriks tripepenjuru bersimetri $\mathbf{A}$ *dalam bentuk*

$$\mathbf{A} = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & b_3 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}.$$

[60 markah]

(b) Hasilkan sifar dalam 2 pemasukkan terakhir vektor

$$\begin{pmatrix} 4 \\ -3 \\ -2 \\ -1 \\ -2 \end{pmatrix},$$

menggunakan

(i) pembalik Householder;

(ii) pemutar Givens.

[40 markah]

2. (a) Compute the orthogonal transformation matrix $\mathbf{Q}$ in the QR factorization of the matrix

$$\mathbf{A} = \begin{pmatrix} 1.000 & 0.500 \\ 0.500 & 0.333 \end{pmatrix}$$

using

(i) the Classical Gram-Schmidt technique;

(ii) Householder reflectors.

Use 3-digit rounding arithmetic in your calculations.

[60 marks]

(b) Compute

(i) $\left\| I - \mathbf{Q}_{CGS}^{T}\mathbf{Q}_{CGS} \right\|_F$

(ii) $\left\| I - \mathbf{Q}_{H}^{T}\mathbf{Q}_{H} \right\|_F$

where $\mathbf{Q}_{CGS}$ is the orthogonal transformation matrix you computed using the classical Gram-Schmidt technique in part a) i) , and, $\mathbf{Q}_H$ is the orthogonal transformation matrix you computed using Householder reflector in part a) ii).

[20 marks]

(c) Use the results in part b) to compare the orthogonalization ability of the classical Gram-Schmidt technique and the Householder reflectors. By this specific example, which method is more stable? Why?

[20 marks]

2. (a) Kira matriks transformasi berortogon $\mathbf{Q}$ *dalam pemfaktoran QR bagi matriks*

$$\mathbf{A} = \begin{pmatrix} 1.000 & 0.500 \\ 0.500 & 0.333 \end{pmatrix}$$

menggunakan

(i) teknik Gram-Schmidt klasikal;

(ii) pembalik-pembalik Householder.

Guna aritmetik 3-digit pembulatan dalam pengiraan anda.

[60 markah]

(b) Kira

(i) $\left\| I - \mathbf{Q}_{CGS}^{T}\mathbf{Q}_{CGS} \right\|_F$

(ii) $\left\| I - \mathbf{Q}_{H}^{T}\mathbf{Q}_{H} \right\|_F$

di mana $\mathbf{Q}_{CGS}$ *ialah matriks transformasi berortogon yang anda kira menggunakan teknik Gram-Schmidt klasikal dalam bahagian a) i) , dan,* $\mathbf{Q}_H$ *ialah matriks transformasi berortogon yang anda kira menggunakan pembalik Householder dalam bahagian a) ii).*

[20 markah]

(c) *Guna keputusan-keputusan dalam bahagian b) untuk membandingkan keupayaan pengortogonan teknik Gram-Schmidt klasikal dan pembalik Householder. Berdasarkan contoh khusus ini, kaedah manakah yang lebih stabil? Mengapa?*

[20 markah]

3. **Use 4-digit rounding arithmetic for all calculations in part a) and part b).** The following algorithm computes l_{ij} for $i, j = 1, 2, \ldots, n$, where $\mathbf{L} = \left[l_{ij} \right]$ is the Cholesky factor of a $n \times n$ matrix $\mathbf{A}$, such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

The Cholesky Algorithm:

Input: An $n \times n$ symmetric positive definite matrix $\mathbf{A}$.

Output: The Cholesky factor $\mathbf{L}$.

For $k = 1, 2, \ldots, n$ do

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

For $i = k+1, \ldots, n$

$$l_{ik} = \frac{1}{l_{kk}} \left(a_{ki} - \sum_{j=1}^{k-1} l_{ij} l_{kj} \right)$$

End

End

(a) Use the Cholesky algorithm above to compute the Cholesky factor for the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1.001 & 1.001 \\ 1 & 1 & 2 \end{pmatrix}.$$

[40 marks]

(b) Use the Cholesky factor in part a) to solve the linear system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{b} = \begin{pmatrix} 3 \\ 3.002 \\ 4 \end{pmatrix}.$$

[40 marks]

(c) The exact solution to $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ and $\mathbf{b}$ are as given in parts a) and b) respectively, is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Compare the 2-norm of the absolute errors for your solution in part b). Relate the quality of your solution with the conditioning of $\mathbf{A}$.

[20 marks]

3. ***Guna aritmetik 4-digit pembulatan untuk semua pengiraan dalam bahagian a) dan b).*** *Algoritma berikut mengira* l_{ij} *untuk* $i, j = 1, 2, \ldots, n$, *di mana* $\mathbf{L} = \left[l_{ij} \right]$ *ialah faktor Cholesky bagi matriks* $n \times n$ $\mathbf{A}$, *sehinggakan* $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

The Cholesky Algorithm:

Input: *Satu matriks positif tentu bersimetri* $n \times n$ $\mathbf{A}$.

Output: *Faktor Cholesky* $\mathbf{L}$.

For $k = 1, 2, \ldots, n$ *do*

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

For $i = k+1, \ldots, n$

$$l_{ik} = \frac{1}{l_{kk}} \left(a_{ki} - \sum_{j=1}^{k-1} l_{ij} l_{kj} \right)$$

End

End

(a) *Guna algoritma Cholesky di atas untuk mengira factor Cholesky bagi matriks berikut*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1.001 & 1.001 \\ 1 & 1 & 2 \end{pmatrix}.$$

(b) *Guna faktor Cholesky dalam bahagian a) untuk menyelesaikan sistem linear* $\mathbf{Ax} = \mathbf{b}$ *di mana*

$$\mathbf{b} = \begin{pmatrix} 3 \\ 3.002 \\ 4 \end{pmatrix}.$$

(40 markah)

(c) *Penyelesaian sebenar bagi* $\mathbf{Ax} = \mathbf{b}$ *di mana* $\mathbf{A}$ *dan* $\mathbf{b}$ *adalah seperti yang diberikan dalam bahagian a) dan b) masing-masing, ialah*

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Bandingkan norma-2 ralat mutlak bagi penyelesaian anda dalam bahagian b). Hubungkaitkan kualiti penyelesaian anda dengan suasana $\mathbf{A}$.

[20 markah]

4. (a) Let

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 2 \\ 4 & 5 & 2 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{pmatrix}.$$

The singular value decomposition of $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T = \begin{pmatrix} 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}.$$

Use the singular value decomposition to

(i) determine

I. the rank of $\mathbf{A}$;

II. $\|\mathbf{A}\|_2$;

III. $\|\mathbf{A}\|_F$;

(ii) find orthonormal bases for

I. the column space of $\mathbf{A}^T$;

II. the nullspace of $\mathbf{A}$;

III. the column space of $\mathbf{A}$;

IV. the nullspace of $\mathbf{A}^T$

(iii) determine the least squares solution of $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{b} = \begin{pmatrix} 18 \\ 18 \\ 18 \\ 18 \end{pmatrix}.$$

[60 marks]

(b) Let $\mathbf{A}$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$.

(i) Let $\mathbf{B} = \mathbf{A}^T\mathbf{A}$. Show that the singular values of $\mathbf{B}$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

(ii) If $\mathbf{x}'$ is the computed solution to $\mathbf{Bx} = \mathbf{c}$ and $\mathbf{x}$ is the exact solution, the inequality

$$\frac{1}{cond_2\ \mathbf{B}} \frac{\|\mathbf{r}\|}{\|\mathbf{c}\|} \leq \frac{\|\mathbf{x} - \mathbf{x}'\|}{\|\mathbf{x}\|} \leq cond_2\ \mathbf{B}\ \frac{\|\mathbf{r}\|}{\|\mathbf{c}\|},$$

where $\mathbf{r} = \mathbf{c} - \mathbf{Bx}$, shows how the relative error compares to the relative residual. Discuss the numerical difficulty of solving the normal equation $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ by referring specifically to the inequality above.

[40 marks]

4. *(a)* *Biar*

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 2 \\ 4 & 5 & 2 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{pmatrix}.$$

Penghuraian nila singular $\mathbf{A}$ *diberikan sebagai*

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T = \begin{pmatrix} 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}.$$

Guna penghuraian nilai singular itu dan

(i) *tentukan*

I. *pangkat* $\mathbf{A}$;

II. $\|\mathbf{A}\|_2$;

III. $\|\mathbf{A}\|_F$;

(ii) *cari asas ortonormal bagi*

I. *ruang lajur* $\mathbf{A}^T$;

II. *ruang nol* $\mathbf{A}$;

III. *ruang lajur* $\mathbf{A}$;

IV. *ruang nol* $\mathbf{A}^T$

(iii) *tentukan penyelesaian kuasa dua terkecil* $\mathbf{Ax}=\mathbf{b}$ *di mana*

$$\mathbf{b}=\begin{pmatrix}18\\18\\18\\18\end{pmatrix}.$$

[60 markah]

(b) *Biar* $\mathbf{A}$ *menjadi suatu matriks* $m\times n$ *dengan nilai-nilai singular* $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_n>0$.

(i) *Biar* $\mathbf{B}=\mathbf{A}^T\mathbf{A}$. *Tunjukkan bahawa nilai-nilai singular* $\mathbf{B}$ *ialah* $\sigma_1^2,\sigma_2^2,\ldots,\sigma_n^2$.

(ii) *Jika* $\mathbf{x}'$ *ialah penyelesaian computer* $\mathbf{Bx}=\mathbf{c}$ *dan* $\mathbf{x}$ *ialah penyelesaian sebenar, ketaksamaan*

$$\frac{1}{cond_2\ \mathbf{B}}\frac{\|\mathbf{r}\|}{\|\mathbf{c}\|}\leq\frac{\|\mathbf{x}-\mathbf{x}'\|}{\|\mathbf{x}\|}\leq cond_2\ \mathbf{B}\ \frac{\|\mathbf{r}\|}{\|\mathbf{c}\|},$$

di mana $\mathbf{r}=\mathbf{c}-\mathbf{Bx}$, *menunjukkan bagaimana ralat relatif berbanding baki relatif. Bincangkan kesukaran berangka untuk menyelesaikan persamaan normal* $\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{b}$ *dengan merujuk khusus kepada ketaksamaan di atas.*

[40 markah]

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