UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
2012/2013 Sidang Akademik

Ogos 2013

MAT 122 – Differential Equations I
[Persamaan Pembezaan I]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** (4) questions.

*[**Arahan**: Jawab **semua empat** (4) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) The differential equation $y' = \dfrac{3x^2 + 4x + 2}{2y - 2}$ has a unique solution through the point $(0, -1)$. True or false? Give your justification using a relevant theorem.

(b) Determine the largest interval on which it is guaranteed by the Existence and Uniqueness Theorem that there exists a solution for the initial value problem $(x-3)y' + \ln x \; y = 2x,\; y(5) = 2$.

(c) Solve the differential equation $e^{3x} y' = xy^2$. Express your answer in the form of $y = \dfrac{A}{e^{-f(x)}[f(x)+1]+B}$, where A, B are constants, and $f(x)$ is a linear function. Determine the values of A and B and the linear function $f(x)$.

[100 marks]

1. (a) Persamaan pembezaan $y' = \dfrac{3x^2 + 4x + 2}{2y - 2}$ *mempunyai penyelesaian unik melalui titik* $(0, -1)$. *Betul atau salah? Berikan penjelasan menggunakan teorem yang sesuai.*

(b) Tentukan selang terbesar di mana teorem kewujudan dan keunikan menjamin bahawa wujudnya penyelesaian unik bagi masalah nilai $(x-3)y' + \ln x \; y = 2x,\; y(5) = 2$.

(c) Selesaikan persamaan pembezaan $e^{3x} y' = xy^2$. *Tuliskan jawapan dalam bentuk* $y = \dfrac{A}{e^{-f(x)}[f(x)+1]+B}$, *di mana* A, B *ialah pemalar, dan* $f(x)$ *ialah suatu fungsi linear. Tentukan nilai A dan B dan fungsi linear* $f(x)$.

[100 markah]

2. (a) (i) Given the differential equation $x^2 y'' - 3xy' + 4y = 0, x \neq 0$, show that $y_1(x) = x^2$ is a solution of the differential equation.

(ii) Find a second solution, $y_2(x)$ of the differential equation.

(iii) Compute the Wronskian, $W\{y_1(x), y_2(x)\}$ and state whether the two solutions are linearly dependent or linearly independent.

(b) (i) Solve the non-homogeneous second order differential equation $y'' - y' - 2y = e^{3x}$ using the method of undermined coefficient.

(ii) Repeat (i) by using the method of variation of parameters.

(iii) Compare the answers obtained in (i) and (ii).

(c) Find a function $N(x, y)$ such that the following differential equation is exact.

$$\left(y^{\frac{1}{2}} x^{-\frac{1}{2}} + \frac{x}{x^2 + y} \right) dx + N(x, y) dy = 0$$

[100 marks]

2. (a) (i) Diberi persamaan pembezaan $x^2 y'' - 3xy' + 4y = 0, x \neq 0$, *tunjukkan bahawa* $y_1(x) = x^2$ *merupakan penyelesaian bagi persamaan pembezaan tersebut.*

(ii) Dapatkan suatu penyelesaian kedua, $y_2(x)$ *untuk persamaan pembezaan tersebut.*

(iii) Hitungkan Wronskian, $W\{y_1(x), y_2(x)\}$ *dan nyatakan sama ada dua penyelesaian tersebut bersandar linear atau tidak bersandar linear.*

(b) (i) Selesaikan persamaan pembezaan tidak homogen peringkat kedua $y'' - y' - 2y = e^{3x}$ *dengan menggunakan kaedah pekali tak ditentukan.*

(ii) Ulangi (i) dengan menggunakan kaedah ubahan parameter.

(iii) Bandingkan jawapan yang diperoleh dalam (i) dan (ii).

(c) Carikan suatu fungsi $N(x, y)$ *supaya persamaan pembezaan yang diberi adalah tepat.*

$$\left(y^{\frac{1}{2}} x^{-\frac{1}{2}} + \frac{x}{x^2 + y} \right) dx + N(x, y) dy = 0$$

[100 markah]

3. (a) (i) Given the initial value problem $y' = 2xy, y(0) = 2$, use Heun's method with $h = 0.1$ to approximate the value of $y(0.2)$.

(ii) Solve the differential equation in (i) analytically and hence, compute the absolute error of the approximation obtained in (i).

(b) Find a series solution for the differential equation $y'' - xy = 0$ with $x_0 = 0$, satisfying the initial conditions $y(0) = 1$ and $y'(0) = 2$.

(c) (i) Find the general solution for the differential equation $\frac{dy}{dx} + f'(x)y = f(x)f'(x)$, where $f(x)$ is a differentiable function.

(ii) Hence or otherwise, solve the differential equation $y' + 2y = 4x$.

[100 marks]

3. (a) (i) Diberi masalah nilai awal $y' = 2xy, y(0) = 2$*, dengan menggunakan kaedah Heun dengan* $h = 0.1$*, anggarkan nilai* $y(0.2)$.

(ii) Selesaikan persamaan pembezaan di (i) secara analisis. Maka, hitungkan ralat mutlak bagi anggaran yang diperoleh di (i).

(b) Dapatkan penyelesaian siri untuk persamaan pembezaan $y'' - xy = 0$ *dengan* $x_0 = 0$*, di mana persamaan pembezaan tersebut memenuhi keadaan nilai awal* $y(0) = 1$ *dan* $y'(0) = 2$.

(c) (i) Dapatkan penyelesaian umum untuk persamaan pembezaan $\frac{dy}{dx} + f'(x)y = f(x)f'(x)$*, di mana* $f(x)$ *ialah suatu fungsi yang dapat dibezakan.*

(ii) Maka, atau sebaliknya, selesaikan persamaan pembezaan $y' + 2y = 4x$.

[100 markah]

4. (a) Show that $e^{\lambda(t-t_0)}\mathbf{v}$, where t_0 is a constant, is a solution for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

(b) Solve the initial-value problem $\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

(c) (i) The rate of growth of a population, $\dfrac{dN}{dt}$ is directly proportional to the population, N, at a certain time, t. The differential equation involved is $\dfrac{dN}{dt} = kN$, where k is a constant whose value is yet to be determined. Given the initial condition $N(0) = N_0$, solve the initial value problem.

(ii) How long would it take for the population to reach twice its starting value? Express your answer in terms of *k*.

[100 marks]

4. *(a) Tunjukkan bahawa* $e^{\lambda(t-t_0)}\mathbf{v}$*, di mana* t_0 *ialah pemalar, merupakan penyelesaian untuk sistem* $\mathbf{x}' = \mathbf{A}\mathbf{x}$*, jika* $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

(b) Selesaikan masalah nilai awal $\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

(c) (i) Kadar pertumbuhan untuk suatu populasi, $\dfrac{dN}{dt}$ *adalah berkadar terus dengan saiz populasi,* N*, pada suatu masa tertentu,* t*. Persamaan pembezaan yang terlibat ialah* $\dfrac{dN}{dt} = kN$*, di mana* k *ialah pemalar yang nilainya belum ditentukan. Diberi nilai awal* $N(0) = N_0$*, selesaikan masalah nilai awal tersebut.*

(ii) Bilakah populasi tersebut akan mencecah nilai dua kali ganda nilai permulaannya? Berikan jawapan anda dalam sebutan k.

[100 markah]

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