UNIVERSITI SAINS MALAYSIA

Second Semester Examination
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MGM 563 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) An experiment is repeated for n trials. The probability of showing outcome 0, 1 or 2 is $P_0,\ P_1$ or P_2 respectively; and $\sum_{i=0}^{2} P_i = 1,\ i = 0,\ 1\ ,\ 2.$ Show that the probability that $1\ \mathit{and}\ 2$ both occur at least once is $1-\ \ 1-P_1\ \ ^{n} -\ \ 1-P_2\ \ ^{n} +\ \ P_0\ \ ^{n}$.

(b) A box contains 15 marbles. 4 of them are black, 6 are white and the remaining are red. 5 marbles are selected randomly one at a time without replacement.
(i) Find the probability of getting the sequence of black, black, red, white, white.
(ii) If the marbles are numbered 1 to 15, find the probability of getting a specific sequence *i.e.* 2, 4, 10, 11, 14.
(iii) Find the probability that each of the marbles will be of the same colour.

(c) The moment generating function of Y is defined by $M_Y\ \ t\ \ = e^{at+(b^2t^2/2)}$. Show that $Var\ \ Y\ \ =\ \ a^2+b^2\ \ -a^2 = b^{2.}$

[20 marks]

1. (a) Satu eksperimen diulangi n *kali. Kebarangkalian untuk menunjukkan kesudahan 0, 1 atau 2 adalah masing-masing* P_0, P_1 *atau* P_2 *; dan* $\sum_{i=0}^{2} P_i = 1,\ i = 0,\ 1\ ,\ 2.$ *Tunjukkan bahawa kebarangkalian bahawa kedua-dua 1 dan 2 muncul sekurang-kurangnya satu kali ialah* $1-\ \ 1-P_1\ \ ^{n} -\ \ 1-P_2\ \ ^{n} +\ \ P_0\ \ ^{n}$.

(b) Satu kotak mengandungi 15 guli. 4 daripadanya berwarna hitam, 6 berwarna putih dan selebihnya berwarna merah. 5 guli dipilih secara rawak satu demi satu tanpa gantian.
(i) Cari kebarangkalian untuk mendapat guli mengikut turutan hitam, hitam, merah, putih, putih.
(ii) Jika guli dinomborkan daripada 1 hingga 15, cari kebarangkalian untuk mendapat turutan tertentu iaitu 2, 4, 10, 11, 14.
(iii) Cari kebarangkalian bahawa setiap guli akan mempunyai warna yang sama

(c) Fungsi penjana momen untuk Y *ditakrifkan sebagai* $M_Y\ \ t\ \ = e^{at+(b^2t^2/2)}$. *Tunjukkan bahawa* $Var\ \ Y\ \ =\ \ a^2+b^2\ \ -a^2 = b^{2.}$

[20 markah]

2. (a) Two coins are flipped. The probability that the first coin shows a head is 0.6 and the probability that a head shown on the second coin is 0.7. Let X be the total number of heads shown.

(i) Find the probability mass function for this experiment when, $X = 0,\ 1$ and 2.

(ii) Derive the cumulative distribution function $F(X)$ and plot the graph.

(iii) Compute $Var(X)$.

(b) The joint probability density function of X and Y is defined by

$f_{X,Y}(x,y) = \frac{1}{2}(x+y),\ 0 \le x \le 1, 0 \le y \le 1.$ Find

(i) the marginal probability density function of X & Y

(ii) $f_{Y|X}(y|x)$

(iii) $E(Y|x)$.

[20 marks]

2. (a) Dua syiling dijentik. Kebarangkalian bahawa syiling pertama menunjukan "kepala" ialah 0.6 dan kebarangkalian "kepala" muncul pada syiling kedua ialah 0.7. Biarkan X sebagai jumlah bilangan "kepala" muncul.

(i) Cari fungsi jisim kebarangkalian untuk eksperimen ini apabila $x = 0,\ 1$ *dan 2.*

(ii) Terbit fungsi taburan longgokan F(X) dan plotkan graf tersebut.

(iii) Kira Var (X).

(b) Fungsi ketumpatan kebarangkalian tercantum X *dan* Y *ditakrifkan sebagai*

$f_{X,Y}(x,y) = \frac{1}{2}(x+y),\ 0 \le x \le 1, 0 \le y \le 1.$ *Cari*

(i) f.k.k sut untuk X & Y

(ii) $f_{Y|X}(y|x)$

(iii) $E(Y|x)$.

[20 markah]

3. (a) Let $X_1, X_2, \ldots, X_n$ be the random variables having a normal distribution $N(\mu, \sigma^2),\ \theta = \sigma^2$ such that $0 < \theta < \infty$ and μ is known.

(i) Write the log likelihood function, $\log(\theta|x)$

(ii) Find the maximum likelihood estimator of θ.

(iii) Show that the maximum likelihood estimator is unbiased, i.e. $E(\hat{\theta}) = \theta$

(b) X & Y are independent random variables from the uniform distribution on the interval $(0,1)$. Find the joint density of $f_{U,V}(u,v)$ given that $U = X + Y$ and $V = X/Y$.

[20 marks]

3. (a) Biarkan $X_1, X_2, \ldots, X_n$ *sebagai pembolehubah rawak yang mempunyai taburan normal* $N(\mu, \sigma^2)$, $\theta = \sigma^2$ *dengan* $0 < \theta < \infty$ *dan* μ *diketahui.*

(i) Tulis fungsi kebolehjadian log, iaitu $\log(\theta | x)$

(ii) Cari penganggar kebolehjadian maksimum θ

(iii) Tunjukkan bahawa penganggar kebolehjadian maksimum adalah saksama iaitu $E(\hat{\theta}) = \theta$

(b) X & Y *adalah pembolehubah rawak tak bersandar daripada taburan seragam pada selang* $(0,1)$. *Cari fungsi ketumpatan kebarangkalian tercantum untuk* $F_{U,V}(u,v)$ *diberi* $U = X + Y$ *dan* $V = X/Y$.

[20 markah]

4 (a) Let $X_1, X_2, \ldots, X_n$ be a random sample with pdf $f(x;\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$ and $\theta > 0$.

(i) What is the likelihood ratio for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$?

(ii) What is the critical region specified by the likelihood ratio test criterion?

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample having a binomial distribution with parameters $b(1, p)$. If $\overline{X}$ is an unbiased estimator of p, find the Cramer Rao lower bound for the variance of unbiased estimator of p.

(c) Let $X_1, X_2, \ldots, X_n$ be a random variables having a normal $N(\theta, 4)$ distribution. The hypotheses tests are

$$H_0 : \theta = 0$$
$$H_1 : \theta > 0$$

The null hypothesis is rejected and the alternative hypothesis is accepted if and only if the observed mean of a random sample of size 25 is greater than or equal to 2/5. Find the power function of this test.

[20 marks]

4. (a) Biarkan $X_1, X_2, \ldots, X_n$ *sebagai suatu sampel rawak dengan fkk* $f(x;\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$ *dan* $\theta > 0$.

(i) Apakah nisbah kebolehjadian untuk menguji $H_0 : \theta = 1$ *lawan* $H_1 : \theta > 1$*?*

(ii) Apakah rantau genting yang dispesifikasikan oleh kriteria ujian nisbah kebolehjadian?

(b) Biarkan $X_1, X_2, \ldots, X_n$ *sebagai suatu sampel rawak yang mempunyai taburan binomial dengan parameter* $b(1, p)$. *Jika* $\overline{X}$ *ialah penganggar saksama untuk* p, *cari batas bawah Cramer Rao untuk varians penganggar saksama* p.

(c) *Biarkan* $X_1, X_2, \ldots, X_n$ *sebagai pembolehubah rawak yang mempunyai taburan normal* $N(\theta, 4)$. *Ujian hipotesis ialah*

$$H_0 : \theta = 0$$

$$H_1 : \theta > 0$$

Hipotesis nol ditolak dan hipotesis alternatif diterima jika dan hanya jika min cerapan bagi sampel rawak bersaiz 25 adalah melebihi atau bersamaan 2/5. Cari fungsi kuasa bagi ujian ini.

[20 markah]

5. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal $N(\mu, 225)$ distribution. The hypotheses tests are

$$H_0 : \mu = 59$$

$$H_1 : \mu \neq 59$$

(i) What is the critical region of size $\alpha = 0.05$ specified by the likelihood ratio test criterion?

(ii) If $n = 100$ and $\overline{X} = 56.13$, what is your test conclusion?

(iii) What is the sample size required if we are 95% confident that the maximum error of the estimate of μ is 1.5?

(b) Y has a binomial distribution with parameters *n* and *p*. We reject $H_0 : p = 1/3$ and accept $H_1 : p > 1/3$ if $Y \geq c$. Find *n* and *c* if the power function of the test $\gamma(p)$ is such that $\gamma(1/3) = 0.10$ and $\gamma(3/4) = 0.95$ approximately.

[20 marks]

5. (a) Biarkan $X_1, X_2, \ldots, X_n$ *sebagai suatu sampel rawak daripada taburan normal* $N(\mu, 225)$.

Ujian hipotesis ialah

$$H_0 : \mu = 59$$

$$H_1 : \mu \neq 59$$

(i) Apakah rantau genting bersaiz $\alpha = 0.05$ *yang dispesifikasikan oleh criteria ujian kebolehjadian?*

(ii) Jika $n = 100$ *dan* $\overline{X} = 56.13$, *apakah kesimpulan ujian anda?*

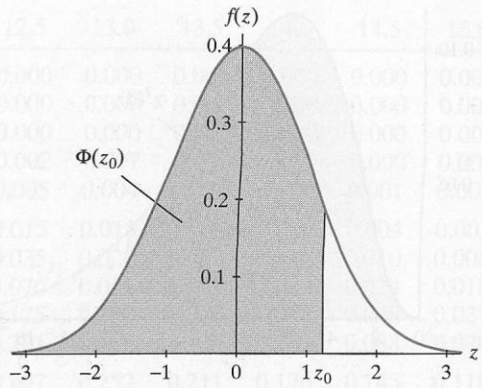
(iii) Apakah saiz sampel yang diperlukan jika kita adalah 95% yakin bahawa ralat maksimum bagi anggaran μ *ialah 1.5?*

(b) Y mempunyai taburan binomial dengan parameter n dan p. Kita menolak $H_0 : p = 1/3$ *dan menerima* $H_1 : p > 1/3$ *jika* $Y \geq c$. *Cari n dan c jika fungsi kuasa bagi ujian* $\gamma(p)$ *adalah* $\gamma(1/3) = 0.10$ *dan* $\gamma(3/4) = 0.95$ *secara hampiran.*

[20 markah]

APPENDIX

The Normal Distribution



$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.240	2.326	2.576	2.807	3.291

Table of Distributions

Distribution	PDF or probability function	mean	variance	MGF
Point mass at a	$I(x=a)$	a	0	e^{at}
Bernoulli(p)	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$pe^t+(1-p)$
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$(pe^t+(1-p))^n$
Geometric(p)	$p(1-p)^{x-1}I(x\geq 1)$	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}\ (t<-\log(1-p))$
Poisson(λ)	$\frac{\lambda^x e^{-x}}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform(a,b)	$I(a<x<b)/(b-a)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal(μ,σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2	$\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$
Exponential(β)	$\frac{e^{-x/\beta}}{\beta}$	β	β^2	$\frac{1}{1-\beta t}\ (t<1/\beta)$
Gamma(α,β)	$\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha\ (t<1/\beta)$
Beta(α,β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
t_ν	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\left(1+\frac{x^2}{\nu}\right)^{(\nu+1)/2}}$	0 (if $\nu>1$)	$\frac{\nu}{\nu-2}$ (if $\nu>2$)	does not exist
χ^2_p	$\frac{1}{\Gamma(p/2)2^{p/2}}x^{(p/2)-1}e^{-x/2}$	p	$2p$	$\left(\frac{1}{1-2t}\right)^{p/2}\ (t<1/2)$

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