UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2011/2012 Academic Session

June 2012

MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Show that $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$ is a probability density function (pdf) of random variable *X*. Then find

(i) $P\left(|X| > \frac{1}{2}\right)$.

(ii) $P(X^2 < 4)$.

[30 marks]

(b) Let *X* be a random variable having pdf $f(x) = 2xe^{-x^2}$, $x \geq 0$.

(i) Find $E(X^2)$.

(ii) Find the pdf of the random variable $Y = X^2$.

[30 marks]

(c) The random variables (*X*, *Y*) have a joint pdf

$$f(x, y) = \frac{1}{8}(x^2 - xy), \quad 0 < x < 2 \text{ and } -x < y < x.$$

(i) Find the conditional pdf of *Y* given *X* = *x*.

(ii) Find $P(Y > 0 \mid X = 1)$.

(iii) Find $E(Y \mid X = 1)$.

[40 marks]

1. (a) Tunjukkan bahawa $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$ *adalah suatu fungsi ketumpatan kebarangkalian (fkk) bagi pembolehubah rawak X. Kemudian cari*

(i) $P\left(|X| > \frac{1}{2}\right)$.

(ii) $P(X^2 < 4)$.

[30 markah]

(b) Biarkan X sebagai suatu pembolehubah rawak dengan fkk $f(x) = 2xe^{-x^2}$, $x \geq 0$.

(i) Cari $E(X^2)$.

(ii) Cari fkk untuk pembolehubah rawak $Y = X^2$.

[30 markah]

(c) Pembolehubah rawak (X, Y) mempunyai fkk tercantum

$$f(x, y) = \frac{1}{8}(x^2 - xy), \quad 0 < x < 2 \text{ dan } -x < y < x.$$

(i) Cari fkk bersyarat untuk Y diberi X = x.

(ii) Cari $P(Y > 0 \mid X = 1)$.

(iii) Cari $E(Y \mid X = 1)$.

[40 markah]

2. (a) Assume that (*X*, *Y*) are continuous random variables having the following joint pdf:

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \ 0 < x < \infty \text{ and } 0 < y < \infty.$$

(i) Find the conditional pdf of *X* given $Y = y$.
(ii) Find the conditional mean of *X* given $Y = y$.
(iii) Find $E(X)$ and $E(XY)$ using the conditional mean of X given $Y = y$, obtained in (ii).
(iv) Find cov(*X*, *Y*).

[50 marks]

(b) Let X_1, X_2, X_3, X_4, X_5 represent independent and identically distributed random variables having pdf $f(x) = 2x$, $0 < x < 1$. Find the probability of the event that three of these random variables have a value greater than $\frac{1}{2}$.

[20 marks]

(c) Let *X* be a positive continuous random variable having distribution function *F* and pdf *f*. Find the pdf, *g*(*y*) of $Y = \frac{1}{1+X}$.

[30 marks]

2. (a) Andaikan bahawa (X, Y) adalah pembolehubah rawak selanjar yang mempunyai fkk tercantum berikut:

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \ 0 < x < \infty \textit{ dan } 0 < y < \infty.$$

(i) Cari fkk bersyarat X diberi $Y = y$.
(ii) Cari min bersyarat X diberi $Y = y$.
(iii) Cari E(X) dan E(XY) dengan menggunakan min bersyarat X diberi $Y = y$, *yang diperoleh dalam (ii).*
(iv) Cari kov(X, Y).

[50 markah]

(b) Biarkan X_1, X_2, X_3, X_4, X_5 *mewakili pembolehubah rawak yang tak bersandar dan bertaburan secaman yang mempunyai fkk* $f(x) = 2x$, $0 < x < 1$. *Cari kebarangkalian peristiwa bahawa tiga daripada pembolehubah rawak ini mempunyai nilai yang lebih besar daripada* $\frac{1}{2}$.

[20 markah]

(c) Biarkan X sebagai pembolehubah rawak selanjar positif yang mempunyai fungsi taburan F dan fkk f. Cari fkk, g(y) bagi $Y = \frac{1}{1+X}$.

[30 markah]

3. (a) Assume that $X_1, X_2, ..., X_n$ represent a random sample from the standard normal distribution. For $m < n$, define $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $\bar{X}_{n-m} = \frac{1}{n-m}\sum_{i=m+1}^{n} X_i$. Find the distribution of the following statistics:

(i) $\bar{X}_m + \bar{X}_{n-m}$.

(ii) $\frac{m(n-m)}{n}\left(\bar{X}_m + \bar{X}_{n-m}\right)^2$.

(iii) $\frac{m\bar{X}_m^2}{(n-m)\bar{X}_{n-m}^2}$.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ represent independent and identically distributed random variables having a Bernoulli distribution with parameter p. Show that $\sum_{i=1}^{n} X_i \sim \text{bin}(n, p)$ using the moment generating function method.

[20 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from an exponential distribution with parameter θ. Find the limiting distribution of

(i) nY_1.

(ii) $n\exp\left(-\theta Y_n\right)$.

[50 marks]

3. (a) Andaikan bahawa $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan normal piawai. Untuk m < n, takrifkan $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i$ dan $\bar{X}_{n-m} = \frac{1}{n-m}\sum_{i=m+1}^{n} X_i$. Cari taburan untuk statistik berikut:

(i) $\bar{X}_m + \bar{X}_{n-m}$.

(ii) $\frac{m(n-m)}{n}\left(\bar{X}_m + \bar{X}_{n-m}\right)^2$.

(iii) $\frac{m\bar{X}_m^2}{(n-m)\bar{X}_{n-m}^2}$.

[30 markah]

(b) Biarkan $X_1, X_2, ..., X_n$ mewakili pembolehubah rawak tak bersandar dan bertaburan secaman yang mempunyai taburan Bernoulli dengan parameter p. Tunjukkan bahawa $\sum_{i=1}^{n} X_i \sim$ bin(n, p) dengan menggunakan kaedah fungsi penjana momen.

[20 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan eksponen dengan parameter* θ. *Cari taburan penghad untuk*

(i) nY_1.

(ii) $n\,\text{eks}(-\theta Y_n)$.

[50 markah]

4. (a) Assume that $X_1, X_2, ..., X_n$ represent a random sample from the Bernoulli distribution having parameter θ, $0 < \theta < 1$.

(i) Find the maximum likelihood estimator for θ.

(ii) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\theta(1-\theta)$.

[40 marks]

(b) Assume that $X_1, X_2, ..., X_n$ is a random sample from the beta, $B(\theta, 1)$ distribution.

(i) Find the method of moments estimator for θ.

(ii) Find the method of moments estimator for the population mean.

[20 marks]

(c) Let $\bar{X}$ and $\bar{Y}$ be the means of two independent samples, each sample of size n and having the $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ distributions, respectively, where the common variance σ^2 is known. Find n so that

$$P\left(\bar{X} - \bar{Y} - \frac{\sigma}{4} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \frac{\sigma}{4}\right) = 0.95.$$

[20 marks]

(d) Assume that $X_1, X_2, ..., X_n$ is a random sample having a $N(\mu, \sigma^2)$ distribution, where μ is known but σ^2 is unknown. Is the random variable $\frac{\bar{X}}{\sigma}$ a pivotal quantity? Explain.

[20 marks]

4. *(a)* *Andaikan bahawa* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan Bernoulli yang mempunyai parameter* θ, $0 < \theta < 1$.

(i) *Cari penganggar kebolehjadian maksimum bagi* θ.

(ii) *Cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama* $\theta(1-\theta)$.

[40 markah]

(b) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan beta,* $B(\theta, 1)$.

(i) *Cari penganggar kaedah momen bagi* θ.

(ii) *Cari penganggar kaedah momen bagi min populasi.*

[20 markah]

(c) *Biarkan* $\bar{X}$ *dan* $\bar{Y}$ *sebagai min dua sampel tak bersandar, setiap sampel dengan saiz n dan masing-masing mempunyai taburan* $N(\mu_1, \sigma^2)$ *dan* $N(\mu_2, \sigma^2)$*, yang mana varians sepunya* σ^2 *adalah diketahui. Cari n supaya*

$$P\left(\bar{X} - \bar{Y} - \frac{\sigma}{4} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \frac{\sigma}{4}\right) = 0.95.$$

[20 markah]

(d) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak yang mempunyai taburan* $N(\mu, \sigma^2)$*, yang mana* μ *diketahui tetapi* σ^2 *tidak diketahui. Adakah pembolehubah rawak* $\frac{\bar{X}}{\sigma}$ *suatu kuantiti pangsian? Jelaskan.*

[20 markah]

5. (a) Let $X_1, X_2, ..., X_n$ be a random sample from the $\exp(\theta)$ distribution with pdf $f(x) = \theta e^{-\theta x}$, $x > 0$, and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean. Find an approximate $100\gamma\%$ confidence interval for θ using the central limit theorem.

[20 marks]

(b) Assume that X is a single observation from a distribution having pdf

$$f(x;\theta) = 2\theta x + 1 - \theta, \quad 0 < x < 1, \quad -1 \le \theta \le 1.$$

(i) Find the most powerful test of size $\alpha = 0.05$ to test $H_0 : \theta = 1$ vs. $H_1 : \theta = 0$.

(ii) For testing $H_0 : \theta \le 0$ vs. $H_1 : \theta > 0$, the following test is used: Reject H_0 if and only if $X \ge \frac{1}{2}$. Find the power function and size of the test.

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ represent a random sample having pdf $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, where $\Theta = \{\theta : \theta > 0\}$. What is the likelihood ratio test of size-α to test $H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$?

[40 marks]

5. (a) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan eks(*θ*) dengan fkk* $f(x) = \theta e^{-\theta x}$, $x > 0$, *dan* $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *sebagai min sampel. Cari suatu selang keyakinan 100*γ*% hampiran untuk* θ *dengan menggunakan teorem had memusat.*

[20 markah]

(b) *Andaikan bahawa X ialah suatu cerapan tunggal daripada taburan yang mempunyai fkk*

$$f(x;\theta) = 2\theta x + 1 - \theta, \quad 0 < x < 1, \quad -1 \le \theta \le 1.$$

(i) *Cari ujian paling berkuasa saiz* α *= 0.05 untuk menguji* $H_0 : \theta = 1$ lawan $H_1 : \theta = 0$.

(ii) *Untuk menguji* $H_0 : \theta \le 0$ lawan $H_1 : \theta > 0$, *ujian berikut digunakan: Tolak* H_0 *jika dan hanya jika* $X \ge \frac{1}{2}$. *Cari fungsi kuasa dan saiz ujian.*

[40 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak yang mempunyai fkk* $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, *yang mana* $\Theta = \{\theta : \theta > 0\}$. *Apakah ujian nisbah kebolehjadian saiz-*α *untuk menguji* $H_0 : \theta \le \theta_0$ *lawan* $H_1 : \theta > \theta_0$*?*

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x) = \frac{1}{N} I_{\{1,2,\ldots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\ldots,n\}}(x)$	np	npq	$\left(q + pe^t\right)^n$
Geometri	$f(x) = pq^x I_{\{0,1,\ldots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}, \; qe^t < 1$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$	λ	λ	$exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}, \; t \neq 0$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\{-(x-\mu)^2 / 2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t + (\sigma t)^2 / 2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}, \; t < \lambda$
Gama	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha, \; t < \lambda$
Khi Kuasa Dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}, \; t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

- ooo O ooo -