UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2011/2012 Academic Session

June 2012

# MAT 263 – Probability Theory
# *[Teori Kebarangkalian]*

Duration : 3 hours
*[Masa : 3 jam]*

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Please check that this examination paper consists of <u>TEN</u> pages of printed materials before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>SEPULUH</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**<u>Instructions</u>**: Answer **<u>all ten</u>** [10] questions.

*[**<u>Arahan:</u>*** *Jawab **<u>semua sepuluh</u>** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].*

1. (a) If $A, B$ and $C$ are mutually independent, is $A$ and $(B \cap C)$ independent? Justify your answer.
   (b) Show that if $P(A) > 0$, then $P(A \cap B \mid A) \geq P(A \cap B \mid A \cup B)$.

[20 marks]

*1. (a) Jika $A, B$ and $C$ adalah saling tak bersandar, adakah $A$ dan $(B \cap C)$ tak bersandar? Tentusahkan jawapan anda.*
   *(b) Tunjukkan bahawa, jika $P(A) > 0$, maka $P(A \cap B \mid A) \geq P(A \cap B \mid A \cup B)$.*

*[20 markah]*

2. A company director believes that workers can be divided into two classes: those who are inclined to take medical leave (MC) and those who are not. The company statistics show that an MC inclined person will be on MC within a year with probability 0.3, whereas the probability decreases to 0.1 for a person who is not an MC inclined person. Assume that 20 percent of the population is inclined to be on MC.
   (a) What is the probability that a new worker will be on MC within his first working year?
   (b) Suppose that a new worker is on MC within his first working year. What is the probability that he is from the group who is inclined on MC?
   (c) What is the conditional probability that a new worker will be on MC in his second working year, given that the worker has been on MC during his first year?

[30 marks]

*2. Seorang pengarah syarikat percaya bahawa pekerja-pekerja boleh dibahagikan kepada dua kelas: mereka yang cenderung untuk mengambil cuti sakit (MC) dan mereka yang tidak. Statistik syarikat menunjukkan bahawa seorang yang cenderung untuk mengambil MC akan mengambil cuti sakit dalam satu tahun dengan kebarangkalian 0.3, manakala kebarangkalian ini menyusut kepada 0.1 bagi seorang yang tidak cenderung mengambil MC. Andaikan bahawa 20 perauts dari populasi adalah cenderung untuk mengambil MC.*
   *(a) Apakah kebarangkalian bahawa seorang pekerja baru akan mengambil MC dalam tahun pertama dia bekerja?*
   *(b) Katakan bahawa seorang pekerja baru mengambil MC dalam tahun pertama dia bekerja. Apakah kebarangkalian bahawa dia adalah dari kumpulan yang cenderung untuk mengambil MC?*
   *(c) Apakah kebarangkalian bersyarat bahawa seorang pekerja baru akan mengambil MC dalam tahun kedua dia bekerja, diberi bahawa pekerja tersebut telah pun mengambil MC semasa tahun pertamanya?*

*[30 markah]*

3. Suppose that a sample of size $n$ is to be chosen randomly without replacement from an urn containing $N$ balls, of which $m$ are white. Let $X$ be the number of white balls selected.
   (a) Derive the expression for the possible values of $N$ .
   (b) If $n = 5, m = 6, N = 10$ and $X_1, \ldots, X_{20}$ are independent random variables,
      (i) use the Markov inequality to obtain a bound on $P\left[\sum_{i=1}^{20} X_i > 15\right]$.
      (ii) use the central limit theorem to approximate $P\left[\sum_{i=1}^{20} X_i > 15\right]$.
   (c) If the experiment is done again but with replacement, do you think the distribution of $X$ stays the same? Explain.

[35 marks]

*3. Katakan bahawa suatu sampel bersaiz $n$ hendak dipilih secara rawak tanpa pengembalian dari suatu bekas yang mengandungi $N$ bola, yang mana $m$ adalah putih. Katakan $X$ adakah bilangan bola putih terpilih.*
   *(a) Terbitkan ungkapan bagi nilai-nilai mungkin $N$ .*
   *(b) Jika $n = 5, m = 6, N = 10$ dan $X_1, \ldots, X_{20}$ adalah pemboleh ubah rawak tak bersandar,*
      *(i) guna ketaksamaan Markov untuk memperolehi suatu batas ke atas $P\left[\sum_{i=1}^{20} X_i > 15\right]$.*
      *(ii) guna teorem had memusat untuk menganggar $P\left[\sum_{i=1}^{20} X_i > 15\right]$.*
   *(c) Jika ujikaji dijalankan semula dengan pengembalian, adakah taburan bagi $X$ masih sama? Terangkan.*

*[35 markah]*

4. A manufacturer is interested in the reliability of the light bulbs it produces. Light bulbs fail if they are not functioning when they are switched on. Suppose that the probability of failure each time the light bulb is switched on is $p$ and the performance is independent from one trial to another.
   (a) The manufacturer is interested in the probability that the light bulb functions at least 50 times. For what values of $p$ is this probability at least 90%?
   (b) At times, when the bulbs fail to function, they are repaired and they are tested again. What is the expected number of trials before a given light bulb fails for the second time?

[30 marks]

4. *Seorang pengeluar berminat dalam kebolehpercayaan lampu mentol yang dikeluarkannya. Lampu mentol gagal jika ianya tidak berfungsi bila dipetik. Katakan bahawa kebarangkalian kegagalan setiap kali lampu dipetik ialah $p$ dan ianya adalah tak bersandar dari satu cubaan ke satu cubaan yang lain.*

   (a) *Pengeluar berminat dengan kebarangkalian bahawa lampu mentol berfungsi sekurang-kurangnya 50 kali. Apakah nilai bagi $p$ supaya kebarangkalian ini adalah sekurang-kurangnya 90%?*

   (b) *Kadangkala, apabila mentol gagal untuk berfungsi, ianya dibaiki dan diuji semula. Berapakah jangkaan bilangan kali cubaan sebelum suatu mentol yang diberi gagal berfungsi untuk kali kedua?*

*[30 markah]*

5. Derive the moment generating function for $X$ which is distributed as normal with mean 6 and variance $\sigma^2$.

[20 marks]

5. *Terbitkan fungsi penjana momen bagi $X$ yang tertabur secara normal dengan min 6 and varians $\sigma^2$.*

*[20 markah]*

6. An R&R station on a PLUS highway has two automated teller machines located next to each other. Customers wishing to obtain cash from one of these machines form a single queue and use the first machine to become available. On average, the service time at the machine 1 on the left is $\theta_1$ seconds, and the service time at the machine 2 on the right (a new machine) is $\theta_2$ seconds. The machines operate on separate power supplies and function independently. Let $S_i$ be the total service times for the machines and $R_i$ be the corresponding remaining service time, $i = 1, 2$. Let $T = \min(R_1, R_2)$ be the waiting time until the first machine becomes available when both machines are in use.

   (a) Find the distribution of $T$.

   (b) If $\theta_1 = 30$ and $\theta_2 = 20$,

      (i) how long should the person at the front of the line expect to wait before a machine becomes available?

      (ii) what is the probability that the person at the front of the line will have to wait more than 15 seconds for a machine to become available?

      (iii) what is the probability that the third person in line will have to wait more than 30 seconds for a machine to become available?

[40 marks]

6. *Suatu stesen R&R di lebuhraya PLUS mempunyai dua mesin ATM yang terletak bersebelahan antara satu sama lain. Pelanggan yang berhasrat untuk mendapatkan tunai daripada salah satu dari mesin tersebut membentuk suatu barisan menunggu tunggal dan menggunakan mesin pertama yang boleh digunakan. Secara purata, masa servis di mesin 1 di sebelah kiri ialah* $\theta_1$ *saat, dan masa servis di mesin 2 di sebelah kanan (suatu mesin baru) ialah* $\theta_2$ *saat. Mesin-mesin beroperasi dengan bekalan kuasa berasingan dan berfungsi secara tak bersandar. Biar* $S_i$ *sebagai jumlah masa servis bagi mesin dan* $R_i$ *sebagai baki masa servis masing-masing,* $i = 1, 2$. *Biar* $T = \min(R_1, R_2)$ *sebagai masa menunggu sehingga mesin pertama tersedia apabila kedua-dua mesin sedang diguna.*
   - *(a) Cari taburan bagi* $T$.
   - *(b) Jika* $\theta_1 = 30$ *dan* $\theta_2 = 20$,
     - *(i) berapa lamakah patut seseorang yang berada di depan barisan, menjangka akan menunggu sebelum sebuah mesin boleh digunakan?*
     - *(ii) apakah kebarangkalian bahawa seseorang di depan barisan akan terpaksa menunggu lebih dari 15 saat bagi menggunakan mesin ATM?*
     - *(iii) apakah kebarangkalian bahawa orang ketiga dalam barisan terpaksa menunggu lebih dari 30 saat bagi menggunakan mesin ATM?*

*[40 markah]*

7. The probability density function for the random variables $X_1$ and $X_2$ is given by

$$f(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if } x_1 + x_2 \le 2,\, x_1 \ge 0,\, x_2 \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

   - (a) Find $f(x_1 \mid x_2)$. Are $X_1$ and $X_2$ independent?
   - (b) Calculate the probability that $X_1 > 2X_2$.
   - (c) Determine the distribution of $X_1 + X_2$.

[40 marks]

7. *Fungsi ketumpatan kebarangkalian bagi pemboleh ubah rawak* $X_1$ *dan* $X_2$ *diberi sebagai*

$$f(x_1, x_2) = \begin{cases} \frac{1}{2} & \textit{jika } x_1 + x_2 \le 2,\, x_1 \ge 0,\, x_2 \ge 0 \\ 0 & \textit{sebaliknya.} \end{cases}$$

   - *(a) Cari* $f(x_1 \mid x_2)$. *Adakah* $X_1$ *dan* $X_2$ *tak bersandar?*
   - *(b) Kira kebarangkalian bahawa* $X_1 > 2X_2$.
   - *(c) Tentukan taburan bagi* $X_1 + X_2$.

*[40 markah]*

8. (a) Suppose that $(X \mid Y = \lambda)$ ~ Gamma $(\alpha_1, \lambda)$ and $Y$ ~ Gamma $(\alpha_2, \beta)$. Find $E(Y \mid X = x)$.

(b) Claims on a newly acquired block of insurance policies are believed to follow an exponential distribution. However, the average claim size is not known with certainty. Prior to receiving any claims, the insurer assumes that the parameter of the exponential distribution has the distribution of Gamma(2,100). The first claim submitted is for RM300. How does this observation affect the insurer's belief about the parameter of the claim size distribution?

[35 marks]

*8. (a) Andaikan bahawa $(X \mid Y = \lambda)$ ~ Gamma $(\alpha_1, \lambda)$ dan $Y$ ~ Gamma $(\alpha_2, \beta)$. Cari $E(Y \mid X = x)$.*

*(b) Tuntutan ke atas suatu polisi insuran blok yang baru diperolehi dipercayai tertabur mengikut suatu taburan eksponen. Walau bagaimana pun, purata saiz tuntutan tidak diketahui dengan pasti. Sebelum menerima sebarang tuntutan, pihak insuran menganggap bahawa parameter bagi taburan eksponen mempunyai taburan Gamma(2,100). Tuntutan pertama yang dihantar ialah bagi RM300. Bagaimanakah cerapan ini memberi kesan terhadap kepercayaan pihak insuran mengenai parameter taburan saiz tuntutan?*

*[35 markah]*

9. Two efficiency experts take independent measurements $Y_1$ and $Y_2$ on the length of time it takes workers to complete a certain task. Each measurement is assumed to have the density function given by

$$f(y) = \begin{cases} \frac{1}{4} y e^{-y/2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the density function for the average $U = (1/2)(Y_1 + Y_2)$ using the transformation method.

(b) Confirm your finding with the method of moment generating function.

[30 marks]

*9. Dua pakar kecekapan mengambil ukuran tak bersandar $Y_1$ and $Y_2$ ke atas tempoh masa yang diambil pekerja untuk menyelesaikan suatu tugas. Setiap ukuran diandaikan mempunytai fungsi ketumpatan yang diberi oleh*

$$f(y) = \begin{cases} \frac{1}{4} y e^{-y/2}, & y > 0 \\ 0, & \text{sebaliknya} \end{cases}$$

*(a) Cari fungsi ketumpatan bagi purata $U = (1/2)(Y_1 + Y_2)$ menggunakan kaedah transformasi.*

*(b) Sahkan jawapan anda dengan kaedah fungsi penjana momen.*

*[30 markah]*

10. The coefficient of variation for a sample of values $Y_1, Y_2, \ldots, Y_n$ is defined by $CV = \dfrac{S}{\overline{Y}}$ where $S$ and $\overline{Y}$ are the standard deviation and the mean of the values, respectively. Let $Y_1, Y_2, \ldots, Y_8$ denote a random sample from a normal distribution with mean $0$ and variance $\sigma^2$. Find the number $c$ such that $P\left(-c \le \dfrac{S}{\overline{Y}} \le c\right) = 0.90$.

[20 marks]

*10. Pekali ubahan bagi suatu sampel dengan nilai* $Y_1, Y_2, \ldots, Y_n$ *ditakrifkan oleh* $CV = \dfrac{S}{\overline{Y}}$ *yang mana* $S$ *dan* $\overline{Y}$ *, masing-masing adalah sisihan piawai dan min bagi nilai-nilai tersebut. Andaikan* $Y_1, Y_2, \ldots, Y_8$ *sebagai suatu sampel rawak dari suatu taburan dengan min* $0$ *dan varains* $\sigma^2$*. Cari suatu nombor* $c$ *supaya* $P\left(-c \le \dfrac{S}{\overline{Y}} \le c\right) = 0.90$.

*[20 markah]*

**APPENDIX /** ***LAMPIRAN***

| ***DISCRETE DISTRIBUTIONS*** | |
|---|---|
| Bernoulli | $f(x) = p^x (1-p)^{1-x}, \quad x = 0,1$<br>$M(t) = 1-p+pe^t$<br>$\mu = p, \quad \sigma^2 = p(1-p)$ |
| Binomial | $f(x) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$<br>$M(t) = (1-p+pe^t)^n$<br>$\mu = p, \quad \sigma^2 = np(1-p)$ |
| Geometric | $f(x) = (1-p)^x p, \quad x = 0,1,2,\ldots$<br>$M(t) = \dfrac{p}{1-(1-p)e^t}, t < \text{In}(1-p)$<br>$\mu = \dfrac{1-p}{p}, \quad \sigma^2 = \mu = \dfrac{1-p}{p^2}$ |
| Negative Binomial | $f(x) = \dfrac{(x+r-p)!}{x!(r-1)!} p^r (1-p)^x, \quad x = 0,1,2,\ldots$<br>$M(t) = \dfrac{p^r}{\left[1-(1-p)e^t\right]^r}, t < -\text{In}(1-p)$<br>$\mu = \dfrac{r(1-p)}{p}, \quad \sigma^2 = \dfrac{r(1-p)}{p^2}$ |
| Poisson | $f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$<br>$M(t) = e^{\lambda(e^t-1)}$<br>$\mu = \lambda, \quad \sigma^2 = \lambda$ |
| Hipergeometric | $f(x) = \dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{t}}, \quad x \le r, x \le n_1, r-x \le n_2,$<br>$\mu = \dfrac{rn_1}{n}, \quad \sigma^2 = \dfrac{rn_1 n_2 (n-r)}{n^2 (n-1)}$ |

| *CONTINUOUS DISTRIBUTION* | |
|---|---|
| Uniform | $f(x)=\frac{1}{b-a},\quad a\le x\le b$ <br> $M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)},\quad t\ne 0,\quad M(0)=1$ <br> $\mu=\frac{a+b}{2},\quad \sigma^2=\frac{(b-a)^2}{12}$ |
| Exponential | $f(x)=\frac{1}{\theta}e^{-x/\theta},\ 0\le x\le\infty$ <br> $M(t)=\frac{1}{1-\theta t},\quad t<1/\theta$ <br> $\mu=\theta,\quad \sigma^2=\theta^2$ |
| Gamma | $f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-x/\theta},0\le x\le\infty$ <br> $M(t)=\frac{1}{(1-\theta t)^{\alpha}},\quad t<1/0$ <br> $\mu=\alpha\theta,\quad \sigma^2=\alpha\theta^2$ |
| Chi Square | $f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2},\quad 0\le x\le\infty$ <br> $M(t)=\frac{1}{(1-2t)^{r/2}},\quad t<\frac{1}{2}$ <br> $\mu=r,\quad \sigma^2=2r$ |
| Normal | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left[(x-\mu)^2/2\sigma^2\right]},-\infty<x<\infty$ <br> $M(t)=e^{\mu t+\sigma^2t^2/2}$ <br> $E(X)=\mu,\quad Var(X)=\sigma^2$ |
| Beta | $f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},0<x<1$ <br> $\mu=\frac{\alpha}{\alpha+\beta},\quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ |

| ***FORMULA*** | |
|---|---|
| 1. | $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ |
| 2. | $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad \lvert r \rvert < 1$ |
| 3. | $\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = \ a+b \ ^n$ |
| 4. | $\sum_{x=0}^{n} \binom{n}{x} \binom{r-n}{r-x} = \binom{n}{r}$ |
| 5. | $\sum_{x=0}^{n} \binom{n+k-1}{k} w^k = \ 1-w \ ^{-n}, \quad \lvert w \rvert < 1$ |
| 6. | $\Gamma \ \alpha \ = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \Gamma \ \alpha \ = \ \alpha - 1 \ !$ |
| 7. | $B \ \alpha, \beta \ = \int_0^1 x^{\alpha-1} \ 1-x \ ^{\beta-1} dx$ |
| 8. | $B \ \alpha, \beta \ = \frac{\Gamma \ \alpha \ \Gamma \ \beta}{\Gamma \ \alpha+\beta}$ |
| 9. | Polar coordinates: $y = r\cos\theta$<br>$z = r\sin\theta$ |

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