UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2011/2012 Academic Session

June 2012

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SIX pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ENAM muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Using properties of matrix algebra, expand and simplify the expression

$$B\ \ -A-3C\ \ +C\ \ A-3B\ \ +3\ \ B^2+C^2$$

and then compute it with the following matrices *A*, *B* and *C*

$$A=\begin{bmatrix} 5 & -3 \\ -14 & 3 \end{bmatrix} \qquad B=\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}.$$

(b) (i) Let A be an $n\times n$ matrix. Prove that if $A^T A = A$, then A is symmetric and $A^2 = A$.

(ii) If *B* and *C* are symmetric matrices, determine the condition for the product *BC* to be symmetric also.

(c) Given

$$\begin{array}{rcrcrcrcrcr} -x_1 & + & x_2 & + & 5x_3 & + & 8x_4 & - & 6x_5 & = & 1 \\ & & -x_2 & - & 2x_3 & - & 3x_4 & + & 4x_5 & = & -2 \\ 2x_1 & & & - & 6x_3 & - & 10x_4 & + & 4x_5 & = & k \end{array}.$$

(i) Find the value of *k* which makes the system consistent.

(ii) Use the value of *k* obtained in (i) to find the general solution to the system of equations and express it in parametric vector form.

(d) Let $A=\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Find the inverse of *A*.

[100 marks]

1. (a) Menggunakan ciri-ciri aljabar matriks, kembangkan dan permudahkan ungkapan

$$B\ \ -A-3C\ \ +C\ \ A-3B\ \ +3\ \ B^2+C^2$$

kemudian hitungkannya dengan matriks A, B dan C yang berikut

$$A=\begin{bmatrix} 5 & -3 \\ -14 & 3 \end{bmatrix} \qquad B=\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}.$$

(b) (i) Biar A sebagai suatu matrix $n\times n$. *Buktikan bahawa jika* $A^T A = A$, *maka A simetrik dan* $A^2 = A$.

(ii) Jika matriks B dan C simetrik, tentukan syarat supaya hasildarab BC juga simetrik.

(c) *Diberi*

$$\begin{array}{rcrcrcrcrcr} -x_1 & + & x_2 & + & 5x_3 & + & 8x_4 & - & 6x_5 & = & 1 \\ & & -x_2 & - & 2x_3 & - & 3x_4 & + & 4x_5 & = & -2 \\ 2x_1 & & & - & 6x_3 & - & 10x_4 & + & 4x_5 & = & k \end{array} .$$

(i) *Cari nilai k yang menjadikan sistem tersebut konsisten.*

(ii) *Guna nilai k yang diperoleh dalam (i) untuk mencari penyelesaian umum bagi sistem tersebut dan nyatakan dalam bentuk berparameter.*

(d) *Biar* $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. *Cari songsangan bagi* A.

[100 markah]

2. (a) Are the vectors $u_1 = (1,2,4,2)$, $u_2 = (3,3,3,9)$ and $u_3 = (-1,1,5,-5)$ linearly independent in $\mathbb{R}^4$? If so, prove it. If not, write a linear dependence relation among them.

(b) Let $V = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y=5z\}$.

(i) Find the set S such that $\mathscr{L}(S) = V$.

(ii) From the set S obtained in (i), form an orthonormal basis for V by using the Gram-Schmidt process.

(iii) Explain why S cannot be a basis for $\mathbb{R}^3$.

(c) Let $W = \{a+bx+cx^2 \in P_2(\mathbb{R}) \mid ab = c^2\}$. Prove that W is a subspace of $P_2(\mathbb{R})$ or use a counterexample to show that W is not a subspace of $P_2(\mathbb{R})$.

(d) Given

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

(i) Determine the basis for its column space.

(ii) Deduce the rank of B, $\rho(B)$ from the answer obtained in (i).

[100 marks]

2. *(a)* *Adakah vektor* $u_1 = (1,2,4,2)$, $u_2 = (3,3,3,9)$ *dan* $u_3 = (-1,1,5,-5)$ *tak bersandar linear dalam* $\mathbb{R}^4$ *? Jika ya, buktikannya. Jika tidak, tuliskan suatu hubungan bersandaran linear di kalangan vektor tersebut.*

(b) *Biar* $V=\{(x,y,z)\in\mathbb{R}^3 \mid x+2y=5z\}$.

(i) *Cari set S sedemikian hingga* $\mathscr{L}(S)=V$.

(ii) *Dari set S yang diperoleh dalam (i), bentukkan suatu asas berortonormal bagi V menggunakan proses Gram-Schmidt .*

(iii) *Terangkan mengapa S tidak boleh menjadi asas* $\mathbb{R}^3$.

(c) *Biar* $W=\{a+bx+cx^2\in P_2(\mathbb{R}) \mid ab=c^2\}$. *Buktikan bahawa W ialah suatu subruang* $P_2(\mathbb{R})$ *atau gunakan contoh lawan bagi menunjukkan yang W bukan subruang* $P_2(\mathbb{R})$.

(d) *Diberi*

$$B=\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}.$$

(i) *Tentukan asas ruang lajurnya.*

(ii) *Deduksikan pangkat B,* $\rho(B)$ *dari jawapan yang diperoleh dalam (i).*

[100 markah]

3. (a) Show that $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $(x,y)T=(x+2y,x-|y|)$ is not a linear transformation.

(b) Let $T:\mathbb{R}^3\to\mathbb{R}^2$ and $S:\mathbb{R}^2\to\mathbb{R}^2$ be linear transformations defined by

$$(x,y,z)T=(x+2y+4z,3x+5y+z)$$
$$(x,y)S=(2x+3y,3x+5y)$$

(i) Is *S* one to one? Justify your answer.

(ii) Find $\operatorname{Im}T$. Is $\operatorname{Im}T=\mathbb{R}^2$? Justify your answer.

(iii) By the dimension theorem, what is dim Ker *T* ? Is *T* one-to-one? Is *T* onto? Justify your answers.

(iv) Find the standard matrix *C* for $T\circ S$ such that $(x,y,z)C=(x,y,z)T\circ S$.

(c) Given that *U* is a subspace of $\mathbb{R}^3$ with basis $\{(1,1,1),(1,4,1)\}$.

(i) Find $U^\perp$, the orthogonal complement of *U*.

(ii) Show that $U\oplus U^\perp=\mathbb{R}^3$.

(d) Suppose that *U* and *W* are subspaces of an inner product space *V*.

(i) Show that $(W^\perp)^\perp=W$.

(ii) If $U\subseteq W$, then $W^\perp\subseteq U^\perp$.

[100 marks]

3. (a) *Tunjukkan bahawa* $T:\mathbb{R}^2 \to \mathbb{R}^2$ *yang tertakrif oleh* $x,y\ T = x+2y, x-|y|$ *bukan suatu transformasi linear.*

(b) *Biar* $T:\mathbb{R}^3 \to \mathbb{R}^2$ *dan* $S:\mathbb{R}^2 \to \mathbb{R}^2$ *transformasi linear yang tertakrif oleh*

$$x,y,z\ T = x+2y+4z, 3x+5y+z$$
$$x,y\ S = 2x+3y, 3x+5y$$

(i) *Adakah S satu-ke-satu? Justifikasikan jawapan anda.*

(ii) *Cari* $\operatorname{Im}T$. *Adakah* $\operatorname{Im}T = \mathbb{R}^2$? *Justifikasikan jawapan anda.*

(iii) *Melalui teorem dimensi, apakah dim Ker T ? Adakah T satu-ke satu? Adakah T menyeluruh? Justifikasikan jawapan anda.*

(iv) *Cari matriks piawai C bagi* $T \circ S$ *sedemikian hingga* $x,y,z\ C = x,y,z\ T \circ S$.

(c) *Diberi U adalah subruang* $\mathbb{R}^3$ *dengan asas* $(1,1,1),(1,4,1)$.

(i) *Cari* $U^{\perp}$, *pelengkap berortogon bagi U.*

(ii) *Tunjukkan bahawa* $U \oplus U^{\perp} = \mathbb{R}^3$.

(d) *Andai U dan W adalah subruang dari suatu ruang hasildarab terkedalam V.*

(i) *Tunjukkan bahawa* $W^{\perp\ \perp} = W$.

(ii) *Jika* $U \subseteq W$, *maka* $W^{\perp} \subseteq U^{\perp}$.

[100 markah]

4. (a) Find the equation $y = mx + c$ of the least squares line that best fits the data points (0,0), (1,-1) and (3,-4).

(b) Let $T:\mathbb{R}^5 \to \mathbb{R}^4$ and $\alpha = e_1, e_2, e_3, e_4, e_5$ is the standard basis of $\mathbb{R}^5$ and $\beta = v_1, v_2, v_3, v_4$ is some basis of $\mathbb{R}^4$. Suppose $e_1\ T = \underline{0}$, $e_2\ T = v_1$, $e_3\ T = 2v_1 + 5v_2$, $e_4\ T = 3v_1 + 6v_2 + 8v_3$ and $e_5\ T = 4v_1 + 7v_2 + 9v_3$.

(i) Find the matrix $T_{\alpha,\beta}$.

(ii) Explain why $x_1, x_2, x_3, x_4, x_5\ _{\alpha} = x_1, x_2, x_3, x_4, x_5$.

(iii) Verify your answer in (i) by showing that $x_1, x_2, x_3, x_4, x_5\ _{\alpha} T_{\alpha,\beta} = x_1, x_2, x_3, x_4, x_5\ T\ _{\beta}$.

(c) (i) Prove that if an $n \times n$ matrix A is non-singular, then 0 is not an eigenvalue of A.

(ii) If $A = A^{-1}$ and λ is an eigenvalue for A, then $\lambda = \pm 1$.

(d) Given

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and the polynomial $f(x) = (x-2)^2(x-1)$ is such that $f(A) = \underline{0}$ (the *characteristic equation* for *A*).

(i) Find the matrices *D*, *C* and C^{-1} such that $CDC^{-1} = A$.

(ii) Find A^n.

[100 marks]

4. *(a)* *Cari persamaan* $y = mx + c$ *of untuk garislurus kuasadua terkecil yang merupakan penghampiran terbaik bagi titik data (0,0), (1,-1) dan (3,-4).*

(b) *Biar* $T : \mathbb{R}^5 \to \mathbb{R}^4$ *dan* $\alpha = \{e_1, e_2, e_3, e_4, e_5\}$ *asas piawai bagi* $\mathbb{R}^5$ *dan* $\beta = \{v_1, v_2, v_3, v_4\}$ *suatu asas bagi* $\mathbb{R}^4$. *Andai* $(e_1)T = \underline{0}$, $(e_2)T = v_1$, $(e_3)T = 2v_1 + 5v_2$, $(e_4)T = 3v_1 + 6v_2 + 8v_3$ *dan* $(e_5)T = 4v_1 + 7v_2 + 9v_3$.

(i) *Cari matriks* $T_{\alpha,\beta}$.

(ii) *Terangkan mengapa* $[(x_1, x_2, x_3, x_4, x_5)]_\alpha = (x_1, x_2, x_3, x_4, x_5)$.

(iii) *Tentusahkan jawapan anda dalam (i) dengan menunjukkan bahawa* $[(x_1, x_2, x_3, x_4, x_5)]_\alpha T_{\alpha,\beta} = [(x_1, x_2, x_3, x_4, x_5)T]_\beta$.

(c) *(i)* *Buktikan bahawa jika A ialah suatu matriks* $n \times n$ *yang tak singular, maka 0 bukan nilai eigen bagi A.*

(ii) *Jika* $A = A^{-1}$ *dan* λ *ialah nilai eigen bagi A, maka* $\lambda = \pm 1$.

(d) *Diberi*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

dan polinomial $f(x) = (x-2)^2(x-1)$ *adalah sedemikian hingga* $f(A) = \underline{0}$ *(persamaan cirian bagi A).*

(i) *Cari matriks D, C dan* C^{-1} *sedemikian hingga* $CDC^{-1} = A$.

(ii) *Cari* A^n.

[100 markah]

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