UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2011/2012

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MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Find the 25th percentile of the distribution having probability density function (pdf)

$$f(x) = \frac{|x|}{4}, \; -2 < x < 2.$$

[20 marks]

(b) Let X have the pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$, that is symmetric at $x = 0$. Why is $E(X)$ not equal to zero?

[20 marks]

(c) Let X and Y have the joint pdf $f(x, y) = 2e^{-(x+y)}$, $0 < x < y < \infty$. Find

(i) the conditional pdf of Y given $X = x$, i.e. $f_{Y|X}(y \mid x)$.

(ii) the conditional mean of Y given $X = x$, i.e. $E(Y|X = x)$.

[40 marks]

(d) Let X_1 and X_2 have the joint pdf $h(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2 < \infty$. Find the joint pdf of $Y_1 = 2X_1$ and $Y_2 = X_2 - X_1$.

[20 marks]

1. (a) Cari persentil ke-25 untuk taburan yang mempunyai fungsi ketumpatan kebarangkalian (fkk)

$$f(x) = \frac{|x|}{4}, \; -2 < x < 2.$$

[20 markah]

(b) Biarkan X mempunyai fkk $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$, *iaitu bersimetri pada x = 0. Kenapa E(X) bukan bersamaan dengan sifar?*

[20 markah]

(c) Biarkan X dan Y mempunyai fkk tercantum $f(x, y) = 2e^{-(x+y)}$, $0 < x < y < \infty$. *Cari*

(i) fkk bersyarat Y diberi X = x, iaitu $f_{Y|X}(y \mid x)$.

(ii) min bersyarat Y diberi X = x, iaitu $E(Y|X = x)$.

[40 markah]

(d) Biarkan X_1 *dan* X_2 *mempunyai fkk tercantum* $h(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2 < \infty$. *Cari fkk tercantum untuk* $Y_1 = 2X_1$ *dan* $Y_2 = X_2 - X_1$.

[20 markah]

2. (a) If T has a t distribution with r degrees of freedom, find the distribution of T^2 using the relevant theorems.

[20 marks]

(b) If X has an F distribution with v_1 and v_2 degrees of freedom, find the distribution of $\frac{1}{X}$ using the relevant theorems.

[20 marks]

(c) Let $X_1, X_2, ..., X_n, ...$ represent a sequence of random variables such that X_n has the distribution function F_n and density function f_n. Find the limiting distribution of random variable X_n, where $X_n \sim N\left(1-\frac{1}{n}, \frac{1}{n}\right)$.

[30 marks]

(d) Let X be a random variable having a uniform distribution with pdf $f(x) = 1$, $0 < x < 1$. A random sample of size 4 is taken from a population having this distribution.

(i) Let $Y_1 < Y_2 < Y_3 < Y_4$ represents the corresponding order statistics of this random sample. Find the joint pdf of Y_1 and Y_4.

(ii) By using the joint pdf of Y_1 and Y_4 in (i), find the probability that the range of this random sample is less than 0.5, i.e. $P(Y_4 - Y_1 < 0.5)$.

[30 marks]

2. *(a) Jika T mempunyai taburan t dengan darjah kebebasan r, cari taburan untuk* T^2 *dengan menggunakan teorem-teorem berkaitan.*

[20 markah]

(b) Jika X mempunyai taburan F dengan darjah kebebasan v_1 *dan* v_2*, cari taburan untuk* $\frac{1}{X}$ *dengan menggunakan teorem-teorem berkaitan.*

[20 markah]

(c) Biarkan $X_1, X_2, ..., X_n, ...$ *mewakili suatu jujukan pembolehubah rawak sedemikian sehingga* X_n *mempunyai fungsi taburan* F_n *dan fungsi ketumpatan* f_n*. Cari taburan penghad untuk pembolehubah rawak* X_n*, yang mana* $X_n \sim N\left(1-\frac{1}{n}, \frac{1}{n}\right)$.

[30 markah]

(d) *Biarkan X sebagai suatu pembolehubah rawak yang mempunyai taburan seragam dengan fkk f(x) = 1, 0 < x < 1. Suatu sampel rawak saiz 4 diambil daripada populasi yang mempunyai taburan ini.*

(i) *Biarkan* $Y_1 < Y_2 < Y_3 < Y_4$ *mewakili statistik tertib yang sepadan untuk sampel rawak ini. Cari fkk tercantum untuk* Y_1 *dan* Y_4.

(ii) *Dengan menggunakan fkk tercantum untuk* Y_1 *dan* Y_4 *dalam (i), cari kebarangkalian bahawa julat untuk sampel rawak ini adalah kurang daripada 0.5, iaitu* $P(Y_4 - Y_1 < 0.5)$.

[30 markah]

3. (a) Consider the following pdf:

$$f(x) = \frac{1}{6\theta^4} x^3 e^{-\frac{x}{\theta}}, \ 0 < x < \infty, 0 < \theta < 1.$$

(i) Let $X_1, X_2, ..., X_n$ represent a random sample taken from a population with the above pdf. Find a complete and sufficient statistic for θ.

(ii) Find the uniformly minimum variance of unbiased estimator (UMVUE) for θ using the Lehmann Scheffe's theorem.

$$\left[\text{Hint} : \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx\right]$$

[30 marks]

(b) Assume that a random sample of size n is drawn from a population having probability mass function (pmf) $f_X(x) = \dfrac{\theta^{2x} e^{-\theta^2}}{x!}$, x = 0, 1, 2, ... Find the maximum likelihood estimator of θ.

[20 marks]

(c) Let $X_1, X_2, ..., X_n$ represent a random sample of size n from the Poisson distribution with pmf $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, ...

(i) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of λ.

(ii) Show that $\hat{\lambda} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ is an efficient estimator for λ.

[30 marks]

(d) Let $X_1, X_2, ..., X_n$ represents a random sample of size n from a normal $N(\mu, \sigma^2)$ distribution.

(i) Show that $\dfrac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma^2}$ is a pivotal quantity.

(ii) Construct a 100γ% confidence interval for σ^2 using the pivotal quantity in (i).

[20 marks]

3. (a) Pertimbangkan fkk berikut:

$$f(x) = \frac{1}{6\theta^4} x^3 e^{-\frac{x}{\theta}}, \; 0 < x < \infty, 0 < \theta < 1.$$

(i) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak yang diambil daripada populasi dengan fkk di atas. Cari statistik cukup dan lengkap untuk θ.

(ii) Cari penganggar saksama bervarians minimum secara seragam (PSVMS) untuk θ dengan menggunakan teorem Lehmann Scheffe.

$$\left[\text{Petua} : \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx\right]$$

[30 markah]

(b) Andaikan bahawa suatu sampel rawak saiz n diambil daripada suatu populasi yang mempunyai fungsi jisim kebarangkalian (fjk) $f_X(x) = \frac{\theta^{2x} e^{-\theta^2}}{x!}$, x = 0, 1, 2, ... Cari penganggar kebolehjadian maksimum untuk θ.

[20 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz n daripada taburan Poisson dengan fjk $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, ...

(i) Cari batas bawah Cramer-Rao untuk varians penganggar saksama λ.

(ii) Tunjukkan bahawa $\hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n}$ adalah penganggar cekap untuk λ.

[30 markah]

(d) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz n daripada taburan normal $N(\mu, \sigma^2)$.

(i) Tunjukkan bahawa $\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma^2}$ adalah suatu kuantiti pangsian.

(ii) Bina suatu selang keyakinan 100γ% untuk σ^2 dengan menggunakan kuantiti pangsian dalam (i).

[20 markah]

4. (a) If $X_1, X_2, ..., X_n$ is a random sample from the gamma distribution, i.e. $G(2, \theta)$ and the sample mean is $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$, find an approximate $100\gamma\%$ confidence interval for θ when n is large.

[30 marks]

(b) Consider a distribution having a pmf $f(x) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$. Let $H_0 : \theta = \dfrac{1}{20}$ and $H_1 : \theta > \dfrac{1}{20}$.

(i) Show that the uniformly most powerful (UMP) test of size α for testing H_0 versus H_1 is given by reject H_0 if and only if $\sum_{i=1}^{n} X_i \geq c$.

(ii) Use the central limit theorem to find a relationship between the constant c in (i) and sample size n of a random sample so that a UMP test of H_0 versus H_1 has a power function $\pi(\theta)$, with an approximate value of $\pi\left(\dfrac{1}{20}\right) = 0.05$.

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample of size n from the normal distribution, $N(\mu, 9)$. Find the likelihood ratio test for testing $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$.

[30 marks]

4. (a) Jika $X_1, X_2, ..., X_n$ ialah suatu sampel rawak daripada taburan gama, iaitu $G(2, \theta)$ dan min sampel ialah $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$, cari selang keyakinan $100\gamma\%$ hampiran untuk θ apabila n adalah besar.

[30 markah]

(b) Pertimbangkan suatu taburan yang mempunyai fjk $f(x) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$. Biarkan $H_0 : \theta = \dfrac{1}{20}$ dan $H_1 : \theta > \dfrac{1}{20}$.

(i) Tunjukkan bahawa ujian paling berkuasa secara seragam (PBS) saiz α untuk menguji H_0 lawan H_1 diberi oleh tolak H_0 jika dan hanya jika $\sum_{i=1}^{n} X_i \geq c$.

(ii) *Gunakan teorem had memusat untuk mencari suatu hubungan antara pemalar c dalam (i) dan saiz sampel n supaya ujian PBS untuk* H_0 *lawan* H_1 *mempunyai fungsi kuasa* $\pi(\theta)$*, dengan nilai hampiran* $\pi\left(\frac{1}{20}\right) = 0.05$*.*

[40 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak saiz n daripada taburan mormal,* $N(\mu, 9)$*. Cari ujian nisbah kebolehjadian untuk menguji* $H_0 : \mu \leq \mu_0$ *lawan* $H_1 : \mu > \mu_0$*.*

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\dots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\,e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\dots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\dots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\,\frac{\lambda^x}{x!}\, I_{\{0,1,\dots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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