UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 514 – Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of ELEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEBELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Explain about the principles that are used in constructing the basic equations of convective heat and mass transfer.

   (b) Discuss briefly on the concept of the boundary layer. Then, state the two-dimensional boundary-layer approximations.

[15 marks]

1. *(a) Terangkan tentang prinsip-prinsip yang digunakan dalam pembinaan persamaan-persamaan asas pemindahan jisim dan haba secara olakan.*

   (b) Bincangkan secara ringkas mengenai konsep lapisan sempadan? Seterusnya, nyatakan penghampiran-penghampiran lapisan sempadan dua dimensi.

[15 markah]

2. Assume that the momentum equation for axisymmetric flow in a circular tube in $x-$ direction is given by equation (1)

$$0=-\frac{dP}{dx}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mu\frac{\partial u}{\partial r}\right), \qquad (1)$$

where the pressure P is independent of r and the dynamic viscosity of the fluid μ is constant.

(a) Derive the following fully developed velocity profile

$$u=\frac{r_s^2}{6\mu}\left(-\frac{dP}{dx}\right)\left(1-\frac{r^2}{r_s^2}\right), \qquad (2)$$

by using the momentum equation (1) subject to the boundary conditions

$$\frac{\partial u}{\partial r}=rx \quad \text{at} \quad r=0, \qquad u=0 \quad \text{at} \quad r=r_s. \qquad (3)$$

(b) What is the velocity gradient near the surface of circular tube?

[Note: $(x,r)=$ axial and radial coordinates of circular tube, respectively; $r_s=$ radius of the circular tube.]

[20 marks]

2. *Anggapkan bahawa persamaan momentum bagi aliran simetri sepaksi dalam tiub membulat dalam arah* x *diberikan oleh persamaan (1)*

$$0 = -\frac{dP}{dx} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \mu \frac{\partial u}{\partial r}\right), \qquad (1)$$

dengan tekanan P *tidak bersandar dengan* r *dan kelikatan dinamik bendalir* μ *adalah malar.*

(a) Terbitkan profil halaju terbangun penuh berikut

$$u = \frac{r_s^2}{6\mu}\left(-\frac{dP}{dx}\right)\left(1 - \frac{r^2}{r_s^2}\right), \qquad (2)$$

dengan menggunakan persamaan momentum (1) tertakluk kepada syarat-syarat sempadan

$$\frac{\partial u}{\partial r} = rx \quad \text{pada} \quad r = 0, \qquad (3)$$
$$u = 0 \quad \text{pada} \quad r = r_s.$$

(b) Apakah kecerunan halaju berdekatan permukaan tiub membulat?

[Nota: $(x, r) =$ *masing-masing adalah koordinat paksian dan jejarian tiub membulat;* $r_s =$ *jejari tiub membulat.]*

[20 markah]

3. Consider a steady flow along a semi-infinite two-dimensional surface with a free stream velocity u_∞ and a free stream temperature T_∞. Let x be measured along the surface and y normal to the surface. Cut out an infinitesimal stationary control volume of unit depth within the boundary layer and consider the various rates of energy transfer across the control surface of the fluid mixture as shown in Figure 1.

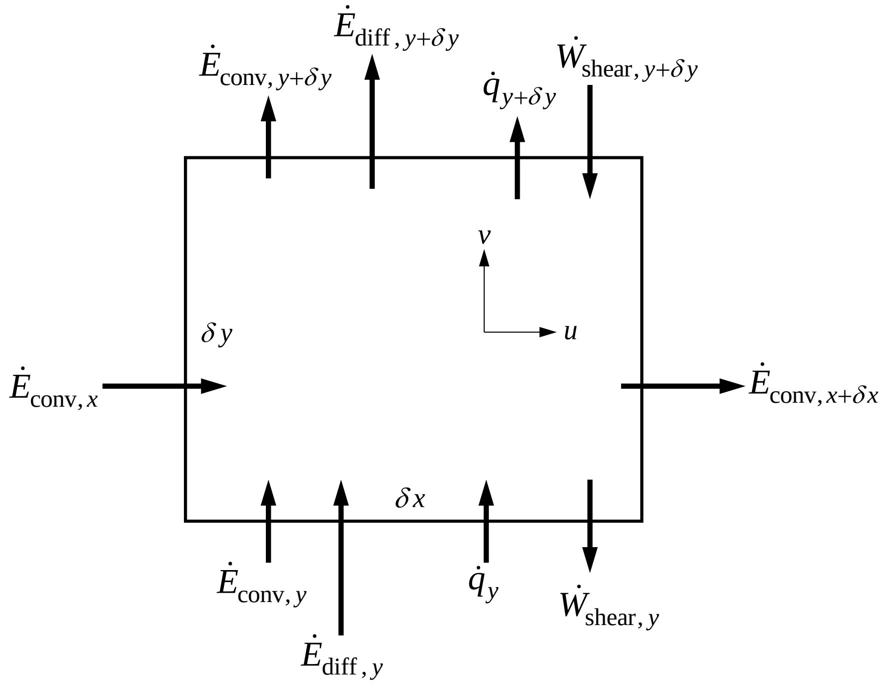


Figure 1: Control volume and energy transfer terms for development of the steady-flow energy differential equation of the boundary layer.

Table 1: Basic rates of energy transfer

$$\dot{E}_{\text{conv},x} = G_x \delta y \left(i + \frac{1}{2} u^2 \right) \quad \text{(convection rate: assuming } u^2 \gg v^2\text{)},$$

$$\dot{E}_{diff,y} = -\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j \right) \delta x \quad \text{(diffusion rate, neglecting Soret effect)},$$

$$\dot{q}_y = -k \left(\frac{\partial T}{\partial y} \right) \delta x \quad \text{(conduction heat transfer, neglecting Dufour effect)},$$

$$\dot{W}_{\text{shear},y} = \ \tau_{yx} u \ \ \delta x \quad \text{(shear force } \times \text{ velocity)}.$$

Based on Figure 1, Table 1 and applying the principle of conservation energy, the two-dimensional boundary-layer approximations, the two-dimensional boundary layer continuity equation $\partial G_x / \partial x + \partial G_y / \partial y = 0$ and the two-dimensional boundary layer momentum equation $(G_x (\partial u / \partial x) + G_y (\partial u / \partial y) = -(dP / dx) + [\partial(\mu(\partial u / \partial y)) / \partial y])$, show that the energy equation of the boundary layer is

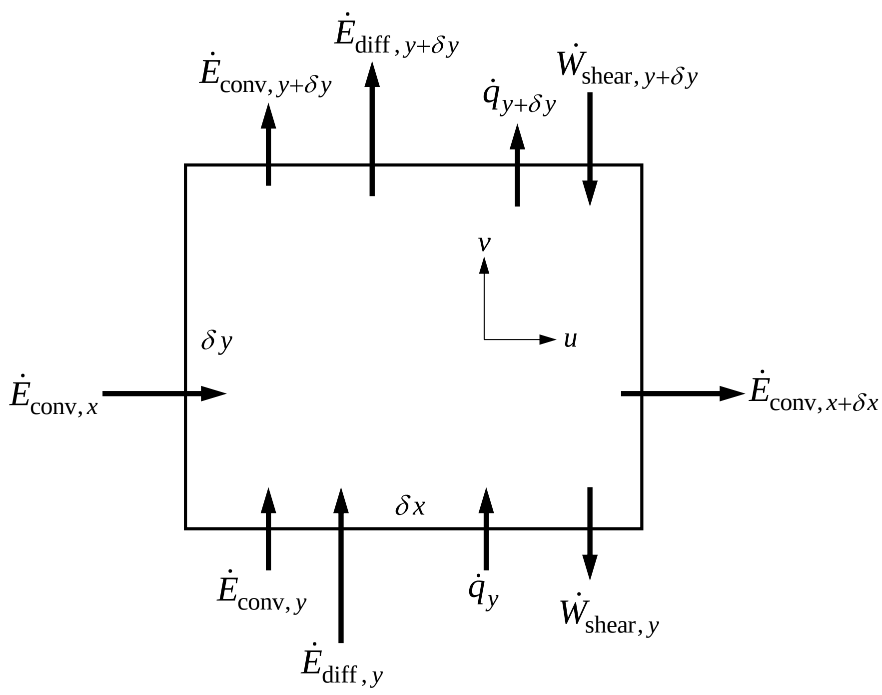
$$G_x \frac{\partial i}{\partial x} + G_y \frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{dP}{dx}. \qquad (4)$$

Then, state the assumption that can be made so that the mass diffusion term in equation (4) can be neglected.

[Note: $(u,v)=$ velocity components along (x,y) axes; $G_x, G_y =$ components of the mass flux vector; $i=$ the mixture enthalpy; $i_j =$ the partial enthalpy of component j; $\gamma_j =$ mass diffusion coefficient for component j in the mixture; $m_j =$ mass concentration of component j in the mixture; $k=$ the thermal conductivity; $T=$ temperature of the fluid mixture; $\tau=$ shear stress.]

[25 marks]

3. *Pertimbangkan suatu aliran mantap terhadap permukaan dua dimensi separuh tak terhingga dengan halaju strim bebas u_∞ dan suhu strim bebas T_∞. x diukur di sepanjang permukaan dan y serenjang terhadap permukaan. Satu unit kedalaman unsur isipadu kawalan pegun dalam lapisan sempadan dikeluarkan dan pertimbangkan beberapa kadar pemindahan tenaga merentasi permukaan kawalan bendalir campuran seperti yang ditunjukkan dalam Rajah 1.*



***Rajah 1**: Isipadu kawalan dan sebutan-sebutan pemindahan haba untuk penerbitan persamaan pembezaan tenaga bagi aliran lapisan sempadan yang mantap.*

***Jadual 1**: Kadar-kadar asas pemindahan tenaga*

$$\dot{E}_{\mathrm{conv},x} = G_x \delta y \left(i + \frac{1}{2} u^2 \right) \quad \text{(kadar olakan: andaikan } u^2 \gg v^2\text{),}$$

$$\dot{E}_{diff,y} = -\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j \right) \delta x \quad \text{(kadar resapan, abaikan kesan Soret),}$$

$$\dot{q}_y = -k \left(\frac{\partial T}{\partial y} \right) \delta x \quad \text{(pemindahan haba konduksi, abaikan kesan Dufour),}$$

$$\dot{W}_{\mathrm{shear},y} = \tau_{yx} u \ \delta x \quad \text{(daya ricih } \times \text{ halaju).}$$

Berdasarkan Rajah 1, Jadual 1 dan aplikasikan prinsip keabadian tenaga, penghampiran-penghampiran lapisan sempadan dua dimensi, persamaan keselanjaran lapisan sempadan dua dimensi $\partial u/\partial x + \partial v/\partial y = 0$ *serta persamaan momentum lapisan sempadan dua dimensi* $(G_x(\partial u/\partial x) + G_y(\partial u/\partial y) = -(dP/dx) + [\partial(\mu(\partial u/\partial y))/\partial y])$ *, tunjukkan bahawa persamaan tenaga lapisan sempadan adalah*

$$G_x \frac{\partial i}{\partial x} + G_y \frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y}\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 + u \frac{dP}{dx}. \qquad (4)$$

Seterusnya, nyatakan andaian yang dibuat supaya sebutan resapan jisim dalam persamaan (4) boleh diabaikan.

[Nota: $(u,v) =$ *komponen-komponen halaju sepanjang paksi* (x,y) *;* $G_x, G_y =$ *komponen-komponen vektor fluks jisim;* $i =$ *entalpi campuran;* $i_j =$ *entalpi separa komponen* j *;* $\gamma_j =$ *pekali resapan jisim komponen* j *dalam campuran;* $m_j =$ *kepekatan jisim komponen* j *dalam campuran;* $k =$ *kekonduksian terma;* $T =$ *suhu bendalir campuran;* $\tau =$ *tegasan ricih.]*

[25 markah]

4. Consider a steady boundary layer flow and heat transfer past a horizontal circular cylinder of radius a in a forced convection flow of a viscous and incompressible fluid of free stream velocity U_∞ and ambient temperature T_∞. It is assumed that the boundary layer approximations are valid. Under these assumptions, the dimensional governing equations are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{5}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{u}_e(\bar{x})\frac{d\bar{u}_e(\bar{x})}{d\bar{x}} + \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{6}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{7}$$

subject to the boundary conditions

$$\begin{aligned} &\bar{u} = \bar{v} = 0, \quad \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q_w}{k} \quad \text{at} \quad \bar{y} = 0, \\ &\bar{u} = \bar{u}_e(\bar{x}), \quad T = T_\infty \quad \text{as} \quad \bar{y} \to y_\infty, \end{aligned} \tag{8}$$

where $\bar{x}$ and $\bar{y}$ are the Cartesian coordinates measured along the cylinder and normal to it, respectively, $\bar{u}$ and $\bar{v}$ are the velocity components along $\bar{x}$ and $\bar{y}$ axes, respectively, $\bar{T}$ is the temperature of the fluid, υ is the kinematic viscosity of the fluid, α is the thermal diffusivity of the fluid, q_w is the constant heat flux, k is the thermal conductivity of the fluid and $\bar{u}_e(\bar{x})$ is the local free stream velocity which is given by $2U_\infty \sin(\bar{x}/a)$. The non-dimensional variables are

$$\begin{aligned} &x = \bar{x}/a, \quad y = \mathrm{Re}^{1/2}(\bar{y}/a), \quad u = \bar{u}/U_\infty, \quad v = \mathrm{Re}^{1/2}(\bar{v}/U_\infty), \\ &\theta = (\bar{T} - T_\infty)/T_\infty, \quad u_e(x) = \bar{u}_e(\bar{x})/U_\infty. \end{aligned} \tag{9}$$

where $\mathrm{Re} = U_\infty a/\upsilon$ is the Reynolds number.

(Note: ψ is the stream function, which is defined in a usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$; take $\alpha = \upsilon/\mathrm{Pr}$ where Pr is the Prandtl number.)

(a) By using non-similar variables of the following form,

$$\psi = xf(x,y), \quad \theta = g(x,y), \tag{10}$$

show that the governing equations (5)-(7) can be reduced to the following system of differential equations

$$\frac{\partial^3 f}{\partial y^3} + f\frac{\partial^2 f}{\partial y^2} + 4\frac{\sin x\cos x}{x} - \left(\frac{\partial f}{\partial y}\right)^2 = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \tag{11}$$

$$\frac{1}{\Pr}\frac{\partial^2 g}{\partial y^2} + f\frac{\partial g}{\partial y} = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \tag{12}$$

with the boundary conditions (8) become

$$\begin{aligned} &f = \frac{\partial f}{\partial y} = 0, \quad \frac{\partial g}{\partial y} = -1 \quad \text{on} \quad y = 0, \\ &\frac{\partial f}{\partial y} = 2\frac{\sin x}{x}, \quad g = 0 \quad \text{as} \quad y \to 0. \end{aligned} \tag{13}$$

(b) Figures 2 and 3 show the numerical results of the system of equations (11) and (12) subject to the boundary conditions (13) in the form of velocity and temperature profiles, respectively, at the lower stagnation point of the cylinder $(x = 0)$ for various values of Prandtl number. Discuss on the obtained results based on these two figures.

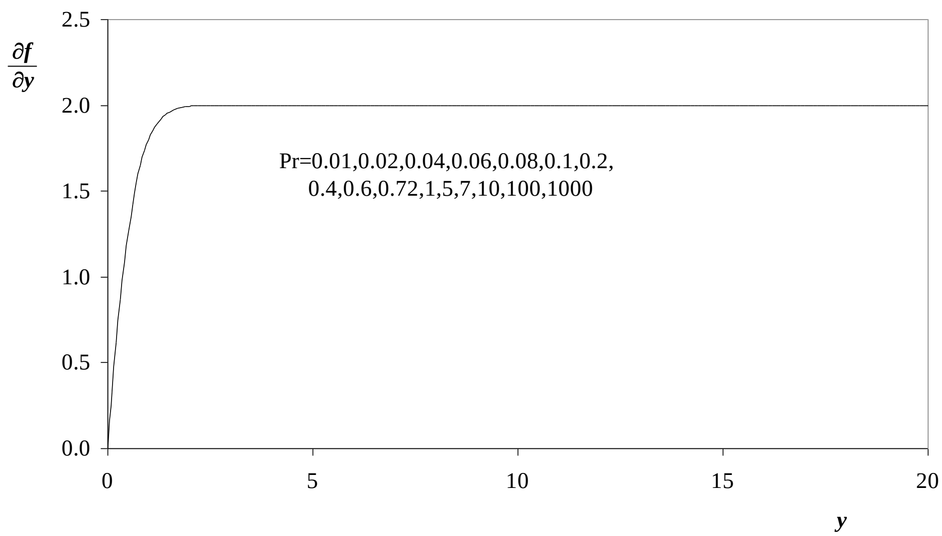


Figure 2: Velocity profile ∂*f*/∂*y* at *x*=0 for various values of Pr

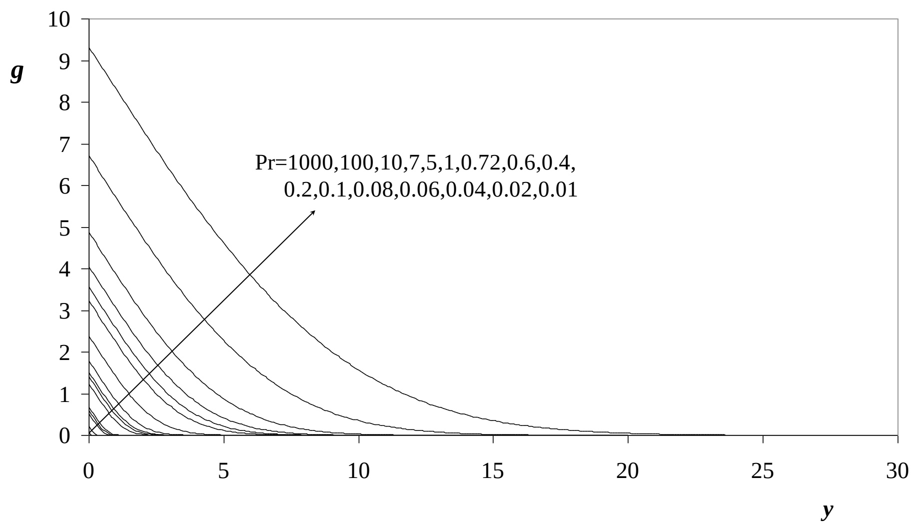


Figure 3: Temperature profiles g at x=0 for various values of Pr

[40 marks]

4. *Pertimbangkan suatu aliran lapisan sempadan dam pemindahan haba yang mantap terhadap silinder membulat mengufuk berjejari* a *dalam aliran olakan paksa bagi bendalir likat dan tak termampat dengan halaju strim bebas* U_∞ *dan suhu persekitaran* T_∞. *Andaikan penghampiran-penghampiran lapisan sempadan adalah sah. Berdasarkan andaian-andaian tersebut, persamaan-persamaan menakluk adalah*

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{5}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{u}_e(\bar{x})\frac{d\bar{u}_e(\bar{x})}{d\bar{x}} + \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{6}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{7}$$

tertakluk kepada syarat-syarat sempadan

$$\begin{aligned} &\bar{u} = \bar{v} = 0, \quad \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q_w}{k} \quad \text{pada} \quad \bar{y} = 0, \\ &\bar{u} = \bar{u}_e(\bar{x}), \quad T = T_\infty \quad \text{apabila} \quad \bar{y} \to y_\infty, \end{aligned} \tag{8}$$

dengan $\bar{x}$ *dan* $\bar{y}$ *masing-masing adalah koordinat-koordinat Cartesian yang diukur di sepanjang silinder dan berserenjang dengannya,* $\bar{u}$ *dan* $\bar{v}$ *masing-masing adalah komponen-komponen halaju pada paksi* $\bar{x}$ *dan* $\bar{y}$ *,* $\bar{T}$ *adalah suhu bendalir,* υ *adalah kelikatan kinematik bendalir,* α *resapan terma bendalir,* q_w *fluks haba malar,* k *adalah kekonduksian terma bendalir dan* $\bar{u}_e(\bar{x})$ *adalah strim halaju bebas setempat yang diberikan oleh* $2U_\infty \sin(\bar{x}/a)$ *. Pemboleh-pemboleh ubah tak berdimensi adalah*

$$\begin{gathered} x = \bar{x}/a, \quad y = \mathrm{Re}^{1/2}(\bar{y}/a), \quad u = \bar{u}/U_\infty, \quad v = \mathrm{Re}^{1/2}(\bar{v}/U_\infty), \\ \theta = (\bar{T} - T_\infty)/T_\infty, \quad u_e(x) = \bar{u}_e(\bar{x})/U_\infty. \end{gathered} \tag{9}$$

dengan $\mathrm{Re} = U_\infty a/\upsilon$ *adalah nombor Reynolds.*

(Nota: ψ *adalah fungsi strim, yang selalunya ditakrifkan sebagai* $u = \partial\psi/\partial y$ *dan* $v = -\partial\psi/\partial x$*; ambil* $\alpha = \upsilon/\mathrm{Pr}$ *dengan Pr adalah nombor Prandtl.)*

(a) Dengan menggunakan pemboleh-pemboleh ubah tak serupa berbentuk berikut

$$\psi = xf(x,y), \quad \theta = g(x,y), \tag{10}$$

tunjukkan bahawa persamaan-persamaan menakluk (5)-(7) boleh terturun kepada sistem persamaan pembezaan berikut

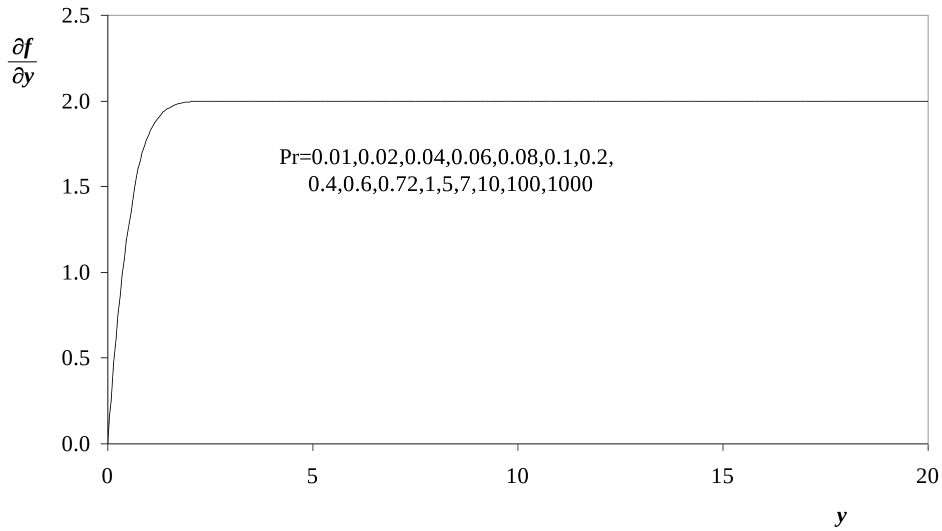
$$\frac{\partial^3 f}{\partial y^3} + f\frac{\partial^2 f}{\partial y^2} + 4\frac{\sin x \,\mathrm{kos}\, x}{x} - \left(\frac{\partial f}{\partial y}\right)^2 = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \tag{11}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 g}{\partial y^2} + f\frac{\partial g}{\partial y} = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \tag{12}$$

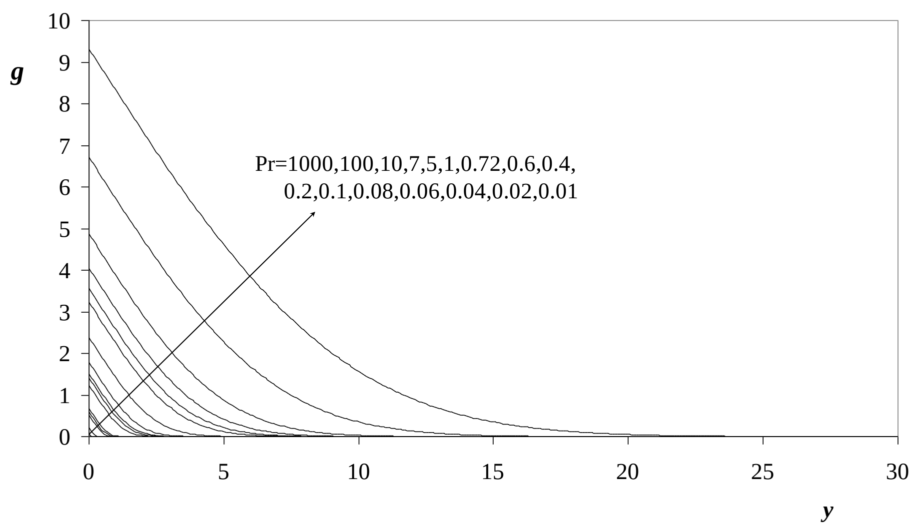
dengan syarat-syarat sempadan (8) menjadi

$$\begin{gathered} f = \frac{\partial f}{\partial y} = 0, \quad \frac{\partial g}{\partial y} = -1 \quad \text{pada} \quad y = 0, \\ \frac{\partial f}{\partial y} = 2\frac{\sin x}{x}, \quad g = 0 \quad \text{apabila} \quad y \to 0. \end{gathered} \tag{13}$$

(b) *Rajah 2 dan 3 masing-masing menunjukkan keputusan berangka bagi sistem persamaan (11) dan (12) tertakluk kepada syarat-syarat sempadan (13) dalam bentuk profil halaju dan suhu, pada titik genangan bawah silinder* $(x = 0)$ *bagi beberapa nilai nombor Prandtl. Bincangkan keputusan yang diperoleh berdasarkan dua rajah tersebut.*



***Rajah 2**: Profil halaju* $\partial f/\partial y$ *pada x=0 bagi beberapa nilai Pr*



***Rajah 3**: Profil suhu g pada x=0 bagi beberapa nilai Pr*

[40 markah]

- ooo O ooo -