UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

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MST 562 – Stochastic Processes
[Proses Stokastik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of <u>SEVEN</u> pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>TUJUH</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

<u>Instructions</u>: Answer **<u>all ten</u>** [10] questions.

*[**<u>Arahan:</u>** Jawab **<u>semua sepuluh</u>** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. A miner is trapped in a mine containing three doors. The first door leads to a tunnel which takes him to safety after two-hours travel. The second door leads to a tunnel which returns him to the mine after three-hours travel and the third door leads to a tunnel which returns him to the mine after four hours travel. Assuming that the miner is at all times equally likely to choose any one of the doors that he has not chosen, what is the expected length of time until the miner reaches safety?

[15 marks]

1. Seorang pekerja lombong terperangkap di dalam lombong yang mempunyai tiga buah pintu. Pintu pertama menuju ke suatu terowong yang membawanya ke tempat selamat selepas dua jam perjalanan. Pintu kedua menuju ke suatu terowong yang membawanya balik ke lombong selepas tiga jam perjalanan, dan pintu ketiga menuju ke suatu terowong yang membawanya balik ke lombong selepas empat jam perjalanan. Dengan andaian bahawa pada semua masa, pekerja lombong tersebut sama berkemungkinan memilih pintu yang belum dipilih sebelumnya, berapakah masa yang dijangka sebelum ia tiba ke tempat selamat?

[15 markah]

2. Let $X_1, X_2, X_3 \ldots$ be a Markov chain on the state space $S = \ 1,2,3$ with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-\alpha & \alpha \\ 1-\alpha & \alpha & 0 \end{bmatrix}, \quad 0 < \alpha < 1.$$

(i) Suppose that the chain is equally likely to be in each of the three states at time 1. Find $P \ X_1 = 2, X_2 = 3, X_3 = 1$.

(ii) Show that P has equilibrium distribution

$$\pi^T = \left(\frac{1-\alpha}{3-\alpha}, \quad \frac{1}{3-\alpha}, \quad \frac{1}{3-\alpha} \right).$$

(iii) Does X_t converge to an equilibrium distribution as $t \to \infty$? Explain your answer.

[15 marks]

2. Andaikan $X_1, X_2, X_3 \ldots$ ialah suatu rantai Markov pada ruang keadaan $S = \ 1,2,3$ dengan matriks peralihan

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-\alpha & \alpha \\ 1-\alpha & \alpha & 0 \end{bmatrix}, \quad 0 < \alpha < 1.$$

(i) Andaikan rantai tersebut mempunyai kemungkinan yang sama untuk berada dalam setiap satu daripada ketiga-tiga keadaan pada masa 1. Dapatkan $P \ X_1 = 2, X_2 = 3, X_3 = 1$.

(ii) Tunjukkan bahawa P mempunyai taburan keseimbangan

$$\pi^T = \left(\frac{1-\alpha}{3-\alpha}, \quad \frac{1}{3-\alpha}, \quad \frac{1}{3-\alpha} \right).$$

(iii) Adakah X_t menumpu ke suatu taburan keseimbangan apabila $t \to \infty$? Jelaskan jawapan anda.

[15 markah]

3. Let $Z_0, Z_1, Z_2, \ldots$ be a process that behaves like a branching process, except for one important difference: The distribution of family size Y_t is not constant for all generations but depends upon the value of Z_t. Specifically, if there are $Z_t > 0$ individuals in generation t, then each of these Z_t individuals has the following family size distribution:

$$Y_t = \begin{cases} 1 & \text{with probability} \quad 1-\frac{1}{Z_t} \\ 0 & \text{with probability} \quad \frac{1}{Z_t} \end{cases}.$$

If $Z_t = 0$, then $Z_{t+1} = 0$ with probability 1. Suppose $Z_0 = 3$, i.e. the process starts with 3 individuals at time 0. Considering $Z_0, Z_1, Z_2, \ldots$ as a Markov chain on the state space $S = \{0, 1, 2, 3\}$, find the transition probability matrix.

[15 marks]

3. Andaikan $\{Z_0, Z_1, Z_2, \ldots\}$ ialah suatu proses yang berkelakuan seperti proses bercabang, kecuali pada suatu perbezaan penting: Taburan saiz keluarga tidak malar bagi semua generasi, tetapi bergantung kepada nilai Z_t. Secara spesifik, jika terdapat $Z_t > 0$ individu dalam generasi t, maka setiap daripada Z_t individu tersebut mempunyai taburan saiz keluarga seperti yang berikut:

$$Y_t = \begin{cases} 0 & \text{dengan kebarangkdian} \quad \left(1-\frac{1}{Z_t}\right) \\ 1 & \text{dengan kebarangkdian} \quad \frac{1}{Z_t} \end{cases}.$$

Jika $Z_t = 0$, maka $Z_{t+1} = 0$ dengan kebarangkalian 1. Andaikan $Z_0 = 3$, iaitu proses tersebut bermula dengan 3 individu pada masa 0. Dengan mempertimbangkan $\{Z_0, Z_1, Z_2, \ldots\}$ sebagai suatu rantai Markov pada ruang keadaan $S = \{0, 1, 2, 3\}$, dapatkan matriks kebarangkalian peralihannya.

[15 markah]

4. Let $X_1, X_2, \ldots$ represent the interarrival times of events of a nonhomogeneous Poisson process having intensity function $\lambda(t)$.

(i) Are the X_i's independent? Explain your answer.

(ii) Are the X_i's identically distributed? Explain your answer.

(iii) Find the distribution of X_1 and the distribution of X_2.

[15 marks]

4. Andaikan $X_1, X_2, \ldots$ mewakili masa antara ketibaan peristiwa-peristiwa suatu proses Poisson tak homogen dengan fungsi intensiti $\lambda(t)$.

(i) Adakah X_i tak bersandar? Terangkan jawapan anda.

(ii) Adakah X_i tertabur secara secaman ? Jelaskan jawapan anda.

(iii) Dapatkan taburan bagi X_1 dan taburan bagi X_2.

[15 markah]

5. Customers arrive at a service facility according to a Poisson process of rate λ customer per hour. Let $N(t)$ be the number of customers that have arrived up to time t. Consider fixed times $0 < s < t$.

(i) Determine the conditional probability $P\{N(t) = n + k \mid N(s) = n\}$.

(ii) Compute the probability $P\{N(s) = 1 \text{ and } N(t) = 3\}$.

(iii) Find $E[N(s).N(t)]$.

(iv) Find $E[N(t) \mid N(s) = n]$.

(v) If S_n denote the time of the n-th arrival, find $E[S_4 \mid N(1) = 3]$.

[20 marks]

5. Pelanggan tiba di suatu kemudahan perkhidmatan menurut suatu proses Poisson dengan kadar λ pelanggan setiap jam. Andaikan N(t) ialah bilangan pelanggan yang telah tiba sehingga masa t. Pertimbangkan masa tetap $0 < s < t$.

(i) Tentukan kebarangkalian bersyarat $P\{N(t) = n + k \mid N(s) = n\}$.

(ii) Hitung kebarangkalian $P\{N(s) = 1 \text{ dan } N(t) = 3\}$.

(iii) Dapatkan $E[N(s).N(t)]$.

(iv) Dapatkan $E[N(t) \mid N(s) = n]$.

(v) Jika S_n *mewakili masa ketibaan yang ke-n, dapatkan* $E[S_4 \mid N(1) = 3]$.

[20 markah]

6. Operations 1, 2 and 3 are performed in succession on a major piece of equipment. Operation k, where $k = 1, 2, 3$, takes a random amount of time T_k, that is exponentially distributed with parameters λ_k and all operations times are independent. Let $X(t)$ denote the operation being performed at time t, with time $t = 0$ marking the start of the first operation. If $\lambda_1 = 5, \lambda_2 = 2$ and $\lambda_3 = 10$, determine $P_n(t) = P\{X(t) = n\}$.

[15 marks]

6. Operasi 1, 2, dan 3 dijalankan secara berturutan ke atas suatu peralatan utama. Operasi k, dengan $k = 1, 2, 3$, memakan masa selama suatu tempoh rawak T_k, yang tertabur secara eksponen dengan parameter λ_k dan semua tempoh masa operasi adalah tak bersandar. Andaikan $X(t)$ mewakili operasi yang dijalankan pada masa t, dengan $t = 0$ menandakan permulaan operasi pertama. Jika $\lambda_1 = 5, \lambda_2 = 2$ dan $\lambda_3 = 10$, tentukan $P_n(t) = P\{X(t) = n\}$.

[15 markah]

7. Consider a queueing system with a single server. Let $X(t)$ denote the number of customers in this system at time *t*. Assume that potential customers arrive according to a Poisson process at rate λ and join the system with probability $1/(k+1)$, where k is the number of customers already in the system. It is assumed that the service time is exponentially distributed with mean $1/\mu$ and is independent of the arrivals.

(i) Let $P_{ij}(h) = P\{X(t+h) = j \mid X(t) = i\}$. Write down $P_{ij}(h)$ for small values of *h*. Specify the birth and death rates for the process $X(t)$.

(ii) Determine the stationary distribution for $X(t)$.

(iii) What is the proportion of time that the system is empty in the long run?

[20 marks]

7. Pertimbangkan suatu sistem giliran dengan pelayan tunggal. Andaikan $X(t)$ *mewakili bilangan pelanggan yang berada dalam sistem pada masa t. Andaikan bakal pelanggan tiba menurut suatu proses Poisson pada kadar* λ *dan menyertai sistem dengan kebarangkalian* $1/(k+1)$, *dengan k sebagai bilangan pelanggan yang telah berada dalam sistem. Masa layan diandaikan tertabur secara eksponen dengan min* $1/\mu$ *dan tak bersandar dengan ketibaan.*

(i) Andaikan $P_{ij}(h) = P\{X(t+h) = j \mid X(t) = i\}$. *Tuliskan* $P_{ij}(h)$ *bagi nilai h kecil. Nyatakan kadar kelahiran dan kadar kematian bagi proses* $X(t)$.

(ii) Tentukan taburan pegun bagi $X(t)$.

(iii) Dalam jangka masa panjang, berapakah kadaran masa sistem tersebut kosong?

[20 marks]

8. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda$ for $i \geq 0$ and death rates $\mu_i = i\mu$ for $i \geq 0$. Assume that $X_0 = 0$. Let m_i be the expected time to go from state *i* to state $i+1$.

(i) Derive the identity $m_i = \dfrac{1}{\lambda_i} + \dfrac{\mu_i}{\lambda_i} m_{i-1}, \quad i \geq 1.$

(ii) Determine the expected time to go from state 2 to state 5.

[15 marks]

8. Pertimbangkan suatu proses kelahiran dan kematian dengan kadar kelahiran $\lambda_i = (i+1)\lambda$ *bagi* $i \geq 0$ *dan kadar kematian* $\mu_i = i\mu$ *bagi* $i \geq 0$. *Andaikan bahawa* $X_0 = 0$. *Andaikan* m_i *adalah masa yang dijangka untuk peralihan dari keadaan i ke keadaan* $i+1$.

(i) Terbitkan identiti $m_i = \dfrac{1}{\lambda_i} + \dfrac{\mu_i}{\lambda_i} m_{i-1}, \quad i \geq 1.$

(ii) Tentukan masa yang dijangka untuk untuk peralihan dari keadaan 2 ke keadaan 5.

[15 marks]

9. Mr. B works on a temporary basis. The mean length of each job he holds is 5 months and the amount of time that he spends between jobs is exponentially distributed with mean 2 months. Let a renewal correspond to the time when he starts a new job.
   (i) Explain why this is a delayed renewal process. What is the mean time between renewals?
   (ii) What is the rate that Mr. B gets a new job in a year?
   (iii) What is the limiting probability that Mr. B is working?

[10 marks]

9. Encik B bekerja secara sambilan. Min tempoh masa setiap pekerjaan yang dipegangnya ialah 5 bulan dan tempoh masa yang dihabiskan antara pekerjaan bertaburan eksponen dengan min 2 bulan. Andaikan suatu pembaharuan bersamaan dengan masa apabila beliau memulakan suatu pekerjaan baru.
   (i) Jelaskan sebab proses ini adalah suatu proses pembaharuan tertunda. Apakah min masa antara pembaharuan?
   (ii) Apakah kadar Encik B mendapat pekerjaan baru dalam setahun?
   (iii) Apakah kebarangkalian penghad bahawa Encik B bekerja?

[10 markah]

10. Write short notes on the following:
   (i) Markovian property
   (ii) transient and recurrent states
   (iii) stationary and independent increment
   (iv) nonhomogeneous Poisson process

[10 marks]

10. Tulis nota pendek mengenai yang berikut:
   (i) sifat Markovian
   (ii) keadaan fana dan keadaan berulang
   (iii) penambahan pegun dan tak bersandar
   (iv) proses Poisson tak homogen

[10 markah]

APPENDIX/LAMPIRAN

1. If X is distributed as Poisson with parameter $\lambda > 0$, then

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0, 1, 2, \cdots$$

2. If X distributed as geometric with parameter $p, \quad 0 < p < 1$, then

$$P(X = x) = p(1-p)^{x-1} \quad ; \quad x = 1, 2, \cdots$$

3. If X distributed as Binomial with parameter $p, \quad 0 < p < 1$, then

$$P(X = x) = \binom{n}{x} p^{x} q^{n-x} \quad ; \quad x = 0, 1, 2, \cdots, n$$

4. If X distributed as exponential with parameter $\lambda > 0$, then

$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

5. If X is distributed as gama with parameter $\alpha > 0$ and $\beta > 0$ then

$$f(x) = \frac{\beta}{\Gamma(\alpha)} (\beta x)^{\alpha-1} e^{-\beta x} \quad ; \quad x > 0$$

6. If X is distributed as normal with parameter μ and $\alpha^2 > 0$ then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad ; \quad -\infty < x < \infty$$

7. Formula of geometric series

$$\sum_{k=0}^{\infty} ar^{k} = \frac{a}{1-r} \quad ; \quad |r| < 1$$

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