
UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

Arahan: Jawab **semua empat** [4] soalan.]

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Let X be a continuous random variable with probability density function (p.d.f.) given by

$$f(x) = \begin{cases} cx^2 & , \quad -2 < x \leq -1 \\ 2c - cx & , \quad -1 < x \leq 0 \\ c/12 & , \quad 0 < x \leq 2 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

- (i) Determine the constant c .
- (ii) Determine the cumulative distribution function (c.d.f.), $F(x)$ and sketch its graph.
- (iii) If X_1 , X_2 and X_3 are three independent observations from X , what is the probability that exactly one of these three numbers is smaller than 0?
- (b) The number of guests check-in at a certain hotel after midnight is a random variable with a Poisson distribution. If the probability of no guests check-in after midnight equals $5/6$, what is the probability that at least one guest check-in after midnight?

[50 marks]

[20 marks]

- (c) The probability density function of X is given by

$$f(x) = \begin{cases} ax^2 + bx + c & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

If $E(X) = 1/2$ and $\text{Var}(X) = 1/4$, find a , b and c .

[30 marks]

1. (a) *Biar X sebagai pembolehubah rawak selang dengan fungsi ketumpatan kebarangkalian diberi oleh*

$$f(x) = \begin{cases} cx^2 & , \quad -2 < x \leq -1 \\ 2c - cx & , \quad -1 < x \leq 0 \\ c/12 & , \quad 0 < x \leq 2 \\ 0 & , \quad \text{di tempat lain.} \end{cases}$$

- (i) *Tentukan pemalar k .*
- (ii) *Tentukan fungsi taburan longgokan, $F(x)$ dan lakarkan grafnya.*
- (iii) *Jika X_1 , X_2 dan X_3 ialah tiga cerapan tak bersandar daripada X , apakah kebarangkalian bahawa tepat satu daripada ketiga-tiga nombor ini lebih kecil daripada 0?*

[50 markah]

- (b) *Bilangan tetamu yang mendaftar masuk di sebuah hotel selepas tengah malam adalah suatu pembolehubah rawak bertaburan Poisson. Jika kebarangkalian tiada tetamu mendaftar masuk selepas tengah malam adalah $5/6$, apakah kebarangkalian bahawa sekurang-kurangnya seorang tetamu mendaftar masuk selepas tengah malam?*

[20 markah]

- (c) *Fungsi ketumpatan kebarangkalian bagi X diberi oleh*

$$f(x) = \begin{cases} ax^2 + bx + c & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{di tempat lain.} \end{cases}$$

Jika $E(X) = 1/2$ dan $\text{Var}(X) = 1/4$, cari a , b dan c .

[30 markah]

2. (a) The number of magazines sold monthly at a certain bookstore is a random variable with expected value 60.

- (i) Give an upper bound to the probability that next month's sales exceed 70.
- (ii) Suppose that the variance of the number of magazines sold monthly is 16. Give a lower bound to the probability that the next month's sales are between 55 and 65 inclusively.

[30 marks]

- (b) If the moment generating function (m.g.f.) of a random variable X is given by

$$M_X(t) = 0.2 + 0.3e^{-2t} + 0.4e^t + 0.1e^{3t},$$

find $P(X \leq 0)$.

[10 marks]

- (c) Suppose that the moment generating function of a random variable X is of the form

$$M_Y(t) = e^{5t + 9t^2/2}.$$

Find the moment generating function of the random variable $Y = 3X + 2$ and use this function to evaluate $E(Y)$.

[30 marks]

- (d) A random variable X has p.d.f. defined by

$$f(x) = \alpha x^{\alpha-1}, \quad 0 < x < 1, \quad \alpha > 0.$$

Find the p.d.f. of the random variable Y where $Y = -2\alpha \ln X$. What is the distribution of Y ?

[30 marks]

2. (a) Bilangan majalah yang dijual di sebuah kedai buku setiap bulan adalah pembolehubah rawak dengan nilai jangkakan 60.

- (i) Beri batas atas kepada kebarangkalian jualan pada bulan hadapan melebihi 70.
- (ii) Andaikan varians bilangan majalah yang dijual setiap bulan adalah 16. Beri batas bawah kepada kebarangkalian jualan pada bulan hadapan adalah antara 55 dan 65.

[30 markah]

- (b) Jika fungsi penjana momen (f.p.m.) suatu pembolehubah rawak X diberi oleh

$$M_X(t) = 0.2 + 0.3e^{-2t} + 0.4e^t + 0.1e^{3t}$$

dapatkan $P(X \leq 0)$.

[10 markah]

- (c) Andaikan bahawa fungsi penjana momen bagi suatu pembolehubah rawak X adalah berbentuk

$$M_Y(t) = e^{5t + 9t^2/2}.$$

Dapatkan fungsi penjana momen bagi pembolehubah rawak $Y = 3X + 2$ dan gunakan fungsi ini untuk menilai $E(Y)$.

[30 markah]

- (d) Pembolehubah rawak X mempunyai f.k.k. yang ditakrif sebagai

$$f(x) = \alpha x^{\alpha-1}, \quad 0 < x < 1, \quad \alpha > 0.$$

Dapatkan f.k.k. bagi pembolehubah rawak Y yang mana $Y = -2\alpha \ln X$.
Apakah taburan Y ?

[30 markah]

3. (a) Suppose that the joint p.d.f. of random variables X and Y is given by

$$f(x, y) = \frac{x-y}{27}, \quad x = 3, 4, 5; \quad y = 0, 1, 2.$$

- (i) Verify that $f(x, y)$ is a joint p.d.f.
- (ii) Find the marginal p.d.f. of X .
- (iii) Compute $P(XY < 5)$, $P(X - Y \geq 3)$ and $P(X = 3|Y = 1)$.

[30 marks]

- (b) Suppose that X and Y are random variables. The marginal p.d.f. of X is

$$f(x) = \begin{cases} kx^2, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Also, the conditional p.d.f. of Y given that $X = x$ is

$$h(y|x) = \begin{cases} \frac{3y^2}{x^3}, & 0 < y < x \\ 0, & \text{elsewhere.} \end{cases}$$

Determine

- (i) the value of k .
- (ii) the marginal p.d.f. of Y . Are X and Y independent?
- (iii) the conditional p.d.f. of X given that $Y = y$.
- (iv) $E X^2 | Y = y$.

[40 marks]

- (c) Let $Z_1, Z_2, \dots, Z_{25}$ be a random sample from the standard normal distribution $N(0,1)$. Let $W = Z_1^2 + Z_2^2 + \dots + Z_{25}^2$. Find $P(9 < W < 36)$.

[10 marks]

- (d) Let $X_1, X_2, X_3, X_4, X_5, X_6$ be a random sample of size 6 from a geometric distribution with $p = 1/4$.
- (i) Find the m.g.f. of $Y = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$.
 - (ii) How is Y distributed?

[20 marks]

3. (a) Andaikan f.k.k. tercantum bagi pembolehubah-pembolehubah rawak X dan Y diberi oleh

$$f(x, y) = \frac{x-y}{27}, \quad x = 3, 4, 5; \quad y = 0, 1, 2.$$

- (i) Sahkan bahawa $f(x, y)$ adalah suatu f.k.k. tercantum.
- (ii) Dapatkan f.k.k. sut bagi X .
- (iii) Hitung $P(XY < 5)$, $P(X - Y \geq 3)$ dan $P(X = 3 | Y = 1)$.

[30 markah]

- (b) Andaikan X dan Y adalah dua pembolehubah rawak. F.k.k. sut bagi X adalah

$$f(x) = \begin{cases} kx^2, & 0 < x < 1 \\ 0, & \text{di tempat lain.} \end{cases}$$

Juga, f.k.k. bersyarat bagi Y diberi $X = x$ adalah

$$h(y|x) = \begin{cases} \frac{3y^2}{x^3}, & 0 < y < x \\ 0, & \text{di tempat lain.} \end{cases}$$

Tentukan

- (i) nilai k .
- (ii) f.k.k. sut bagi Y . Adakah X dan Y tidak bersandar?
- (iii) f.k.k. bersyarat bagi X diberi $Y = y$.
- (iv) $E X^2 | Y = y$.

[40 markah]

- (c) Biar $Z_1, Z_2, \dots, Z_{25}$ menjadi suatu sampel rawak daripada taburan normal piawai $N(0,1)$. Biar $W = Z_1^2 + Z_2^2 + \dots + Z_{25}^2$. Cari $P(9 < W < 36)$.

[10 markah]

- (d) Biar $X_1, X_2, X_3, X_4, X_5, X_6$ sebagai sampel rawak bersaiz 6 daripada taburan geometri dengan $p = 1/4$.

(i) Dapatkan f.p.m. bagi $Y = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$.

(ii) Bagaimana Y tertabur?

[20 markah]

4. (a) Let X and Y have a bivariate normal distribution with parameters

$$\mu_X = 20, \mu_Y = 50, \sigma_X^2 = 9, \sigma_Y^2 = 25, \rho = 0.8.$$

Compute

(i) $E(Y|X=18)$.

(ii) $Var(Y|X=18)$.

(iii) $P(Y \leq 40|X=18)$.

[20 marks]

- (b) Suppose that X and Y are random variables with $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$. If $Y = a + bX$, $a \neq 0$ and $b \neq 0$, show that $\rho_{XY} = 1$ if $b > 0$ and $\rho_{XY} = -1$ if $b < 0$.

[20 marks]

- (c) $X_1, X_2, \dots, X_{25}$ is a random sample of size $n = 25$ from the normal distribution with mean 20 and variance 16. The statistics $\bar{X}$ and S^2 are defined as

$$\bar{X} = \frac{\sum_{i=1}^{25} X_i}{25} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{25} (X_i - \bar{X})^2}{24},$$

respectively.

Find $P(15 < \bar{X} < 25, S^2 < 9)$.

[20 marks]

- (d) Let X, Y be two independent standard normal random variables. Show that a random variable $W = X/Y$ has a Cauchy distribution with p.d.f. defined as

$$f(w) = \frac{1}{\pi (1+w^2)}, \quad -\infty < w < \infty.$$

[40 marks]

4. (a) Biarkan X dan Y mempunyai taburan bivariat normal dengan parameter-parameter

$$\mu_X = 20, \mu_Y = 50, \sigma_X^2 = 9, \sigma_Y^2 = 25, \rho = 0.8.$$

Hitung

- (i) $E(Y|X=18)$.
- (ii) $\text{Var}(Y|X=18)$.
- (iii) $P(Y \leq 40|X=18)$.

[20 markah]

- (b) Andaikan bahawa X dan Y ialah pembolehubah-pembolehubah rawak dengan $\sigma_X^2 > 0$ dan $\sigma_Y^2 > 0$. Jika $Y = a + bX$, $a \neq 0$ dan $b \neq 0$, tunjukkan bahawa $\rho_{XY} = 1$ jika $b > 0$ dan $\rho_{XY} = -1$ jika $b < 0$.

[20 markah]

- (c) $X_1, X_2, \dots, X_{25}$ ialah suatu sampel rawak bersaiz $n = 25$ daripada taburan normal dengan min 20 and varians 16. Statistik $\bar{X}$ dan statistik S^2 ditakrifkan masing-masing sebagai

$$\bar{X} = \frac{\sum_{i=1}^{25} X_i}{25} \quad \text{dan} \quad S^2 = \frac{\sum_{i=1}^{25} (X_i - \bar{X})^2}{24}.$$

Dapatkan $P \ 15 < \bar{X} < 25, S^2 < 9$.

[20 markah]

- (d) Biar X, Y sebagai dua pembolehubah rawak normal piawai yang tak bersandar. Tunjukkan bahawa suatu pembolehubah rawak $W = X/Y$ mempunyai taburan Cauchy dengan f.k.k. ditakrif sebagai

$$f(w) = \frac{1}{\pi (1 + w^2)}, \quad -\infty < w < \infty.$$

[40 markah]

Appendix

	Probability Density Function
Bernoulli	$p^x(1-p)^{1-x}, \quad x=0,1, \quad 0 < p < 1$
Binomial	$\binom{n}{x} p^x(1-p)^{n-x}, \quad x=0,1,\dots,n, \quad 0 < p < 1$
Hypergeometric	$\frac{\binom{n_1}{x} \binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}, \quad x=0,1,\dots, r \leq n \text{ or } x=1,2,\dots, n_1 \leq r$
Geometric	$(1-p)^{x-1} p, \quad x=1,2,\dots$
Negative Binomial	$\binom{x-1}{r-1} p^r(1-p)^{x-r}, \quad x=r, r+1,\dots$
Poisson	$\frac{e^{-\lambda} \lambda^x}{x!}, \quad x=0,1,2,\dots, \quad \lambda > 0$
Uniform	$\frac{1}{\beta-\alpha}, \quad \alpha < x < \beta$
Normal	$\frac{1}{\sqrt{2\pi} \sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty$
Exponential	$\lambda e^{-\lambda x}, \quad x \geq 0$
Gamma	$\frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0, \quad \alpha > 0$
Chi-square	$\frac{1}{\Gamma(r/2) 2^{r/2}} x^{r/2-1} e^{-x/2}, \quad x \geq 0$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha > 0, \quad \beta > 0$