UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 203 – Vector Calculus
[Kalkulus Vektor]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SIX pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ENAM muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Find an equation of the plane in $\mathbb{R}^3$ that passes through the points $A(2,1,1)$, $B(-1,-1,10)$ and $C(1,3,-4)$

[7 marks]

(b) Find the parametric equation for the line through point B and perpendicular to the plane in part (a).

[4 marks]

(c) A second plane passes through point $(2,0,4)$ and has normal vector $\langle 2,-4,-3 \rangle$. Show that this plane and the plane (a) intersect at the point $(2,0,4)$ and the acute angle between them is approximately 43^0.

[8 marks]

1. (a) Dapatkan persamaan satah dalam $\mathbb{R}^3$ *yang melalui titik-titik* $A(2,1,1)$, $B(-1,-1,10)$ *dan* $C(1,3,-4)$.

[7 markah]

(b) Dapatkan persamaan parametrik untuk garis lurus yang melalui titik B *dan serenjang kepada satah di bahagian (a).*

[4 markah]

(c) Suatu satah kedua melalui titik $(2,0,4)$ *dan mempunyai vektor normal* $\langle 2,-4,-3 \rangle$. *Tunjukkan bahawa satah ini dan satah (a) bersilang pada titik* $(2,0,4)$ *dan anggaran sudut di antara mereka ialah* 43^0.

[8 markah]

2. (a) A particle starts moving at the origin with initial velocity $\mathbf{i}-\mathbf{j}+3\mathbf{k}$. Its acceleration is $\mathbf{a}(t)=6t\mathbf{i}+12t^2\mathbf{j}-6t\mathbf{k}$. Find the particle position function at the time t.

[8 marks]

(b) The position vector of an object in $\mathbb{R}^3$ is

$\mathbf{r}(t)=(a\cos\theta t)\mathbf{i}+(a\sin\theta t)\mathbf{j}+\theta^2 t\mathbf{k}$, a constant

Show that if sum of the objects tangential and normal component of acceleration equal half its speed, then the angle $\theta=\dfrac{a}{\sqrt{4a^2-1}}$.

[8 marks]

(c) Let $\mathbf{r}(t)$ be a smooth vector function in $\mathbb{R}^3$ and $\mathbf{T}$ is the unit tangent vector with property $\mathbf{r}' = |\mathbf{r}'|\mathbf{T}$. Show that $\mathbf{r}'' = |\mathbf{r}'|'\,\mathbf{T} + \kappa|\mathbf{r}'|^2\,\mathbf{N}$ and hence show that $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ where κ is the curvature of the curve described by $\mathbf{r}(t)$ at any point on curve.

[6 marks]

2. *(a) Suatu zarah mula bergerak dari asalan dengan halaju awal* $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. *Kecepatannya ialah* $\mathbf{a}(t) = 6t\mathbf{i} + 12t^2\mathbf{j} - 6t\mathbf{k}$. *Dapatkan fungsi kedudukan zarah ini pada masa t.*

[8 markah]

(b) Vektor kedudukan suatu objek dalam $\mathbb{R}^3$ *ialah*

$\mathbf{r}(t) = (a\cos\theta t)\mathbf{i} + (a\sin\theta t)\mathbf{j} + \theta^2 t\mathbf{k}$, *a pemalar*

Tunjukkan bahawa jika hasil tambah komponen tangen kecepatan dan komponen normal kecepatan bersamaan separuh lajunya, maka sudut $\theta = \dfrac{a}{\sqrt{4a^2 - 1}}$

[8 markah]

*(c) Katakan **r**(t) suatu fungsi vektor yang licin dalam* $\mathbb{R}^3$ *dan* $\mathbf{T}$ *ialah vektor unit tangendengan dengan sifat* $\mathbf{r}' = |\mathbf{r}'|\mathbf{T}$. *Tunjukkan bahawa* $\mathbf{r}'' = |\mathbf{r}'|'\,\mathbf{T} + \kappa|\mathbf{r}'|^2\,\mathbf{N}$ *dan seterusnya tunjukkan bahawa* $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ *yang mana* κ *ialah kelengkungan lengkung yang diterangkan oleh* $\mathbf{r}(t)$ *pada sebarang titik di atas lengkung.*

[6 markah]

3. (a) Find the angle of intersection between the helix $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t^2\,\mathbf{j} - t\,\mathbf{k}$ and the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1, 0, 0)$.

[6 marks]

(b) Let $f(x, y) = x^2 - xy + y^2 - y$. Find the direction $\mathbf{u}$ and the value of the derivative in the direction of $\mathbf{u}$, $D_u f$ for the following,

(i) $D_u f(1,-1)$ is largest.

(ii) $D_u f(1,-1)$ is smallest.

(iii) $D_u f(1,-1) = -3$

[9 marks]

(c) Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1,1,2)$.

[8 marks]

3. *(a) Dapatkan sudut persilangan di antara helik* $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t^2\,\mathbf{j} - t\,\mathbf{k}$ *dan lengkung* $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ *pada titik* $(1,0,0)$.

[6 markah]

(b) Katakan $f(x,y) = x^2 - xy + y^2 - y$. *Dapatkan arah* ***u*** *and nilai terbitan pada arah* ***u***, $D_u f$ *untuk berikut,*

(i) $D_u f(1,-1)$ *terbesar.*

(ii) $D_u f(1,-1)$ *terkecil.*

(iii) $D_u f(1,-1) = -3$

[9 markah]

(c) Tunjukkan bahawa elipsoid $3x^2 + 2y^2 + z^2 = 9$ *dan sfera* $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ *adalah tangen antara satu sama lain pada titik* $(1,1,2)$

[8 markah]

4. (a) Let C be the circle $x^2 + y^2 = a^2$, $a > 0$. Show that at each point of curve C, the vector field $\mathbf{F}(x,y) = \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}$ in $\mathbb{R}^2$ is parallel to the line tangent to C at that point.

[6 marks]

(b) Use the Divergence Theorem to find the outward flux of $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2 z)\mathbf{j} + (4x^2 y^3)\mathbf{k}$ across the boundary of the region cut from the first octant by cylinder $x^2 + y^2 = 4$ and the plane $z = 3$.

[9 marks]

4. (a) *Katakan C suatu bulatan* $x^2 + y^2 = a^2$, $a > 0$. *Tunjukkan bahawa pada setiap titik di lengkung C, medan vector* $\mathbf{F}(x, y) = \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}$ *dalam* $\mathbb{R}^2$ *adalah selari dengan garis tangen kepada C pada titik itu.*

[6 markah]

(b) *Gunakan Teorem Divergen untuk mendapatkan fluks keluar bagi* $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2 z)\mathbf{j} + (4x^2 y^3)\mathbf{k}$ *merentasi sempadan kawasan yang dipotong oleh selinder* $x^2 + y^2 = 4$ *dan satah* $z = 3$ *dalam oktan pertama.*

[9 markah]

5. (a) Given a vector forms of Green's Theorem $\oint_C \mathrm{F}\cdot\mathbf{n}\, ds = \iint_D \operatorname{div} \mathrm{F}(x,y)\, dA$, C is a positively oriented, piecewise-smooth and simple close curve in a plane, D is the region bounded by C and $\mathbf{n}$ is the outward unit normal vector on the curve C. Prove that,

$$\iint_D f\ \nabla^2 g\ dA = \oint_C f\,(\nabla g)\cdot\mathbf{n}\ ds - \iint_D \nabla f \cdot \nabla g\ dA$$

[7 marks]

(b) Evaluate

$$\int_C x e^{-2x} dx + (x^4 + 2x^2 y^2) dy$$

where C is the boundary of semi-annular region D in the upper half-plane between circle $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$.

[7 marks]

(c) Use Stoke Theorem to evaluate the flux of the curl of the field $\mathbf{F} = 2z\,\mathbf{i} + 3x\,\mathbf{j} + 5y\,\mathbf{k}$ across the surface

$S:\ \mathbf{r}(r,\theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + (4 - r^2)\mathbf{k}$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$
in the direction of the outward unit normal $\mathbf{n}$,

[7 marks]

5. (a) *Diberikan suatu vektor berbentuk Green's Theorem* $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x,y)\, dA$, *C ialah suatu lengkung berorentasi positif, licin-cebis demi cebis dan tertutup mudah pada suatu satah, D ialah kawasan yang dibatasi oleh C dan* **n** *ialah vektor unit normal atas lengkung. Buktikan bahawa,*

$$\iint_D f\ \nabla^2 g\ dA = \oint_C f\,(\nabla g)\cdot\mathbf{n}\ ds - \iint_D \nabla f \cdot \nabla g\ dA$$

[7 markah]

(b) *Nilaikan*

$$\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy$$

yang mana C ialah sempadan bagi kawasan semiannular D di antara bulatan $x^2 + y^2 = 4$ *dan* $x^2 + y^2 = 1$ *pada separuh bahagian atas satah.*

[7 markah]

(c) *Gunakan Teorem Stoke untuk menilaikan fluk lingkaran suatu medan* $\mathbf{F} = 2z\,\mathbf{i} + 3x\,\mathbf{j} + 5y\,\mathbf{k}$ *yang merentasi permukaan*

$S:\ \mathbf{r}(r,\theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + (4 - r^2)\,\mathbf{k},\ 0 \le r \le 2$ dan $0 \le \theta \le 2\pi$
pada arah unit normal **n** *keluar.*

[7 markah]

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