UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 517 – Computational Linear Algebra
[Aljabar Linear Pengkomputeran]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of <u>SIX</u> pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>ENAM</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

<u>Instructions</u>: Answer **<u>all four</u>** [4] questions.

*[**<u>Arahan:</u>** Jawab **<u>semua empat</u>** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (i) Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $n \times n$ matrix and $\mathbf{b}$ $n \times 1$ vector. The right hand side term of the system experiences a small perturbation $\delta\mathbf{b}$. If $\mathbf{A}$ is nonsingular and $\mathbf{b} \neq \mathbf{0}$, it can be show that

$$\frac{\|\boldsymbol{\delta}\mathbf{b}\|}{\|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|\mathbf{b}\|} \leq \frac{\|\boldsymbol{\delta}\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \frac{\|\boldsymbol{\delta}\mathbf{b}\|}{\|\mathbf{b}\|},$$

where $\delta\mathbf{x}$ is the resulting error in the solution $\mathbf{x}$ due to the perturbation. Discuss the effect on $\mathbf{x}$ when

(a) $\|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ is small.

(b) $\|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ is large.

[40 marks]

(ii) Consider the following linear systems.

System I

$$x_1 + 2x_2 = 3$$
$$3x_1 + 4x_2 = 7$$

System II

$$x_1 + 2x_2 = 3$$
$$2x_1 + 3.999x_2 = 5.999$$

(a) System I has the exact solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and experience a perturbation of $\delta\mathbf{b} = \begin{pmatrix} 0.001 \\ 0.001 \end{pmatrix}$. Solve the **perturbed system** and compute the relative perturbation and the relative error.

(b) System II has the exact solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and experience a perturbation of $\delta\mathbf{b} = \begin{pmatrix} 0.000 \\ 0.001 \end{pmatrix}$. Solve the **perturbed** system and compute the relative perturbation and the relative error.

(c) Which system is ill-conditioned? Confirm by computing the condition numbers of the two systems.

[60 marks]

1. (i) Pertimbangkan sistem linear $\mathbf{Ax} = \mathbf{b}$ *dengan* $\mathbf{A}$ *matriks* $n \times n$ *dan* $\mathbf{b}$ *vektor* $n \times 1$. *Sebutan di sebelah kanan system ini mengalami usikan kecil* $\delta\mathbf{b}$. *Jika* $\mathbf{A}$ *tak singular dan* $\mathbf{b} \neq \mathbf{0}$, *boleh ditunjukkan bahawa*

$$\frac{\|\boldsymbol{\delta}\mathbf{b}\|}{\|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|\mathbf{b}\|} \leq \frac{\|\boldsymbol{\delta}\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \frac{\|\boldsymbol{\delta}\mathbf{b}\|}{\|\mathbf{b}\|},$$

dengan $\delta\mathbf{x}$ *sebagai ralat penyelesaian* $\mathbf{x}$ *yang terhasil akibat daripada usikan tersebut. Bincangkan kesan terhadap penyelesaian* $\mathbf{x}$ *apabila*

(a) $\|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ *kecil.*

(b) $\|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ *besar.*

[40 markah]

(ii) Pertimbangkan sistem linear berikut.

Sistem I

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 + 4x_2 &= 7 \end{aligned}$$

Sistem II

$$\begin{aligned} x_1 + \quad 2x_2 &= 3 \\ 2x_1 + 3.999x_2 &= 5.999 \end{aligned}$$

(a) Sistem I mempunyai penyelesaian tepat $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *dan mengalami usikan* $\delta\mathbf{b} = \begin{pmatrix} 0.001 \\ 0.001 \end{pmatrix}$. *Selesaikan* ***sistem yang terusik*** *dan kira usikan relatif dan ralat relatif.*

(b) Sistem II mempunyai penyelesaian tepat $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *dan mengalami usikan* $\delta\mathbf{b} = \begin{pmatrix} 0.000 \\ 0.001 \end{pmatrix}$. *Selesaikan* ***sistem yang terusik*** *dan kira usikan relatif dan ralat relatif.*

(c) Sistem manakah yang bersuasana tak sihat? Sahkan dengan mengira nombor suasana kedua-dua sistem itu.

[60 markah]

2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

(i) Compute the singular value decomposition of $\mathbf{A}$.

[60 marks]

(ii) Find a particular least squares solution to the system $\mathbf{Ax} = \mathbf{b}$.

[30 marks]

(iii) Are there other solutions? If so, write the general formula for the solutions.

[10 marks]

2. Biar

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \textit{ dan } \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

(i) Kira penghuraian nilai singular $\mathbf{A}$.

[60 markah]

(ii) Cari penyelesaian kuasa dua terkecil yang khusus kepada sistem $\mathbf{Ax} = \mathbf{b}$.

[30 markah]

(iii) Adakah wujud penyelesaian lain? Jika ada, tuliskan rumus am penyelesaian-penyelesaian tersebut.

[10 markah]

3. (i) Describe the QR factorization of a 3×2 matrix $\mathbf{A}$ based on the orthogonalizing techniques below:

(a) The classical Gram-Schmidt technique;
(b) Householder matrices;
(c) Givens matrices.

[60 marks]

(ii) Give a brief discussion on the numerical properties of classical Gram-Schmidt technique which makes it less favorable compared to the Householder and Givens matrices.

[10 marks]

(iii) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix}.$$

Find the QR factorization of $\mathbf{A}$ using the classical Gram-Schmidt technique.

[30 marks]

3. (i) *Huraikan pemfaktoran QR matriks* 3×2 $\mathbf{A}$ *, berdasarkan teknik-teknik pengortogonan di bawah:*
   (a) teknik Gram-Schmidt klasik;
   (b) matrik Householder;
   (c) matriks Givens.

(60 markah)

(ii) *Beri perbincangan ringkas tentang ciri-ciri berangka teknik Gram-Schmidt klasik yang menyebabkan ia kurang digemari berbanding dengan matriks Householder dan Givens.*

[10 markah]

(iii) *Biar*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix}.$$

Cari pemfaktoran QR $\mathbf{A}$*, menggunakan teknik Gram-Schmidt klasik.*

[30 markah]

4. (i) Prove that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $\mathbf{A}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the corresponding eigenvectors, then $\lambda_1 - \sigma, \lambda_2 - \sigma, \ldots, \lambda_n - \sigma$ are eigenvalues of $\mathbf{A} - \sigma\mathbf{I}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the corresponding eigenvectors.

[20 marks]

(ii) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 10 & 0 \\ 1 & 9.8 \end{pmatrix}$$

(a) Perform 3 steps of the Power method to compute an estimate for the largest eigenvalue of $\mathbf{A}$.

(b) Apply a shift of $\sigma = 9.5$ and perform 3 steps of the shifted Power method on matrix $\mathbf{A}$. Compute an estimate for the largest eigenvalue of $\mathbf{A}$ and compare with the results you obtain in part a). Which method gives you a better estimate?

(c) The Power method applied on matrix $\mathbf{A}$ results in a slow convergence. Why?

[80 marks]

4. (i) *Buktikan, jika* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *adalah nilai-nilai eigen* $\mathbf{A}$ *dan* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *adalah vektor-vektor eigen yang sepadan, maka* $\lambda_1 - \sigma, \lambda_2 - \sigma, \ldots, \lambda_n - \sigma$ *adalah nilai-nilai eigen* $\mathbf{A} - \sigma\mathbf{I}$ *dan* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *adalah vektor-vektor eigen yang sepadan.*

[20 markah]

(ii) *Pertimbangkan matriks*

$$\mathbf{A} = \begin{pmatrix} 10 & 0 \\ 1 & 9.8 \end{pmatrix}.$$

(a) *Laksana 3 langkah kaedah Power untuk mengira anggaran kepada nilai eigen* $\mathbf{A}$ *yang terbesar.*

(b) *Guna anjakan* $\sigma = 9.5$ *dan laksana 3 langkah kaedah Power teranjak ke atas* $\mathbf{A}$*. Kira anggaran kepada nilai eigen* $\mathbf{A}$ *yang terbesar dan bandingkan dengan keputusan yang anda perolehi dalam bahagian a). Kaedah manakah memberikan anda anggaran yang lebih baik?*

(c) *Penggunaan kaedah Power ke atas* $\mathbf{A}$ *menghasilkan penumpuan yang perlahan. Mengapa?*

[80 markah]

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