UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Show that the determinant of the matrix $A = \begin{bmatrix} 1 & 1+k & 1 \\ 2+k & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}$ is $-9k^2 - 30k + 96.$

(b) Find values of real number k for which the set $\left\{ \begin{bmatrix} 1 \\ 2+k \\ 1 \end{bmatrix}, \begin{bmatrix} 1+k \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 9 \end{bmatrix} \right\}$ is linearly dependent.

(c) Determine the reduced row echelon form of the matrix $B = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}$.

(d) Find the bases for the row space, the column space and the null space of the matrix $B = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}$.

(e) State the relationship between the rank and the nullity of the matrix in part (d).

(f) Show that the null space of B^T in part (d) is the orthogonal complement of the column space of B.

[100 marks]

1. (a) *Tunjukkan bahawa penentu matriks* $A = \begin{bmatrix} 1 & 1+k & 1 \\ 2+k & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}$ *ialah* $-9k^2 - 30k + 96.$

(b) *Dapatkan nilai-nilai nombor nyata k, yang dengannya set* $\left\{ \begin{bmatrix} 1 \\ 2+k \\ 1 \end{bmatrix}, \begin{bmatrix} 1+k \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 9 \end{bmatrix} \right\}$ *adalah bersandar secara linear.*

(c) *Tentukan bentuk eselon baris terturun matriks* $B = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}.$

(d) *Dapatkan asas ruang baris, asas ruang lajur dan asas ruang nol bagi matriks* $B = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 5 & -10 \\ 1 & 7 & 9 \end{bmatrix}.$

(e) *Nyatakan hubungan di antara pangkat dan kenolan matriks dalam bahagian (d).*

(f) *Tunjukkan bahawa ruang nol* B^T *dalam bahagian (d) adalah pelengkap berortogon ruang lajur* B.

[100 markah]

2. Let $L:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Show that L is a linear transformation.

(b) Is L one to one and onto? Justify your answer.

(c) Find a basis for kernel of L.

(d) Extend the basis in part (c) to obtain a basis for $\mathbb{R}^3$.

(e) Find a basis for range of L.

(f) Based on your result in parts (c), (d) and (e), verify that dim domain L = dim ker L + dim range L.

[100 marks]

2. *Andaikan* $L:\mathbb{R}^3 \to \mathbb{R}^3$ *ditakrifkan oleh*

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Tunjukkan bahawa L adalah suatu transformasi linear.

(b) Adakah L satu dengan satu dan keseluruh? Justifikasikan jawapan anda.

(c) Dapatkan asas bagi inti L.

(d) Kembangkan asas dalam bahagian (c) untuk memperoleh asas bagi $\mathbb{R}^3$.

(e) Dapatkan asas bagi julat L.

(f) Berdasarkan keputusan yang telah kamu perolehi dalam bahagian (c), (d) dan (e), tentusahkan bahawa matra domain L = matra inti L + matra julat L.

[100 markah]

3. (a) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

(i) Verify that the function $(\mathbf{u}, \mathbf{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$ satisfies the properties of an inner product on $\mathbb{R}^2$.

(ii) Use the definition in part (i) to find the length of $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

(iii) State the Cauchy-Schwarz inequality for the Euclidean space $\mathbb{R}^2$ with the inner product defined in part (i).

(iv) Choose an ordered basis S for $\mathbb{R}^2$ and find the matrix of the inner product with respect to S.

(b) Let W be the set of all 3×3 matrices of the form $\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$. Show that W is a subspace of M_{33}, a vector space of all 3×3 matrices where the operation $\oplus$ is the standard addition of matrices and the operation $\odot$ is the standard multiplication of a matrix by a real number.

[100 marks]

3. (a) Andaikan $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *dan* $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *adalah vektor dalam* $\mathbb{R}^2$.

(i) Tentusahkan bahawa fungsi $(\mathbf{u}, \mathbf{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$ *memenuhi sifat-sifat hasil darab terkedalam pada* $\mathbb{R}^2$.

(ii) Guna takrif dalam bahagian (i) untuk mencari panjang $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

(iii) Nyatakan ketaksamaan Cauchy-Schwarz bagi ruang Euklidan $\mathbb{R}^2$ *dengan hasil darab terkedalam yang ditakrifkan dalam bahagian (i).*

(iv) Pilih suatu asas bertertib S bagi $\mathbb{R}^2$ *dan dapatkan matriks hasil darab terkedalam terhadap S.*

(b) *Andaikan* W *adalah set kesemua matriks* 3×3 *berbentuk* $\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$.

Tunjukkan bahawa W *adalah suatu subruang bagi* M_{33}, *suatu ruang vektor kesemua matriks* 3×3 *dengan operasi* $\oplus$ *adalah penambahan piawai matriks dan operasi* $\odot$ *adalah pendaraban piawai matriks dengan nombor nyata.*

[100 markah]

4. (a) Let A be an $n\times n$ matrix and let $B = P^{-1}AP$ be similar to A. Show that if $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ of A, then $P^{-1}\mathbf{x}$ is an eigenvector of B associated with the eigenvalue λ of B.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(i) Find the characteristic polynomial, the eigenvalues, and associated eigenvectors of A.

(ii) Is A similar to a diagonal matrix? Justify your answer.

(iii) If your answer in part (ii) is yes, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

(iv) Is P in part (iii) unique? Justify your answer.

(v) Find the eigenvalues of A^{-1}.

(vi) Find the eigenvalues and associated eigenvectors of A^2.

[100 marks]

4. (a) *Andaikan A adalah suatu matriks* $n \times n$ *dan andaikan* $B = P^{-1}AP$ *serupa dengan A. Tunjukkan bahawa jika* $\mathbf{x}$ *adalah vektor eigen A yang dikaitkan dengan* λ, *nilai eigen A, maka* $P^{-1}\mathbf{x}$ *adalah vektor eigen B yang dikaitkan dengan* λ, *nilai eigen B.*

(b) *Andaikan* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(i) *Dapatkan polinomial cirian, nilai-nilai eigen, dan vektor-vektor eigen yang berkaitan bagi matriks A.*

(ii) *Adakah A serupa dengan suatu matriks pepenjuru? Justifikasikan jawapan anda.*

(iii) *Sekiranya jawapan anda dalam bahagian (ii) adalah ya, dapatkan suatu matriks tak singular P, supaya dengannya* $P^{-1}AP$ *adalah matriks pepenjuru.*

(iv) *Adakah P dalam bahagian (iii) unik? Justifikasikan jawapan anda.*

(v) *Dapatkan nilai-nilai eigen* A^{-1}.

(vi) *Dapatkan nilai-nilai eigen dan vektor-vektor eigen berkaitan bagi* A^2.

[100 markah]

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