UNIVERSITI SAINS MALAYSIA

First Semester Examination
2011/2012 Academic Session

January 2012

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SIX pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ENAM muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Consider the heat equation

$$u_t = \alpha^2 u_{xx}.$$

Suppose we evaluate the solution at the grid point (x_i, t_{n+1}).

(a) Write the FTCS scheme for this case (Remember t_{n+1}).

(b) Is this an implicit or explicit scheme?

(c) By using the Fourier method, obtain the amplication factor for this scheme. Explain why the scheme is unconditionally stable.

[100 marks]

1. Pertimbangkan persamaan haba

$$u_t = \alpha^2 u_{xx}.$$

Andaikan kita menilaikan penyelesaian di titik grid (x_i, t_{n+1}).

(a) Tulis skema FTCS untuk kes ini (Ingat t_{n+1} *).*

(b) Adakah ini skema tersirat atau tak tersirat?

(c) Dengan menggunakan kaedah Fourier, dapatkan faktor amplifikasi untuk skema ini. Terangkan mengapa skema ini stabil tanpa syarat.

[100 markah]

2. Consider the equation $u_t + au_x = 0$ where a is a positive constant.

   (a) Write down the forward time centered space scheme for this equation.

   (b) Do a consistency analysis of this scheme.

   (c) The amplification factor for this scheme is $\lambda = 1 - \sqrt{-1}\, C \sin\theta$ where $C = u\Delta t/\Delta x$. Explain why the scheme is unconditionally unstable.

   (d) State the Lax Equivalence Theorem. Is the above scheme convergent ?

[100 marks]

2. Pertimbangkan persamaan $u_t + au_x = 0$ di mana a ialah pemalar positif.

   (a) Tulis skema beza ke depan terhadap masa dan beza pusat untuk ruang bagi persamaan ini.

   (b) Jalankan analisis kekonsistenan untuk skema ini.

   (c) Faktor amplikasi untuk skema ini ialah $\lambda = 1 - \sqrt{-1}\, C \sin\theta$ dengan $C = u\Delta t/\Delta x$. Terangkan mengapa skema ini tak stabil tanpa syarat.

   (d) Tuliskan teorem kesetaraan Lax. Adakah skema di atas menumpu ?

[100 markah]

3. (a) Consider the following boundary value problem

$$\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 3, 0 < y < 3;$$
$$u(0, y) = 0, u(3, y) = 0,$$
$$\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{3}{4}, \frac{\partial u}{\partial y}\Big|_{y=3} = \frac{5}{4}.$$

Given $\Delta x = \Delta y = 1$, discretize the above PDE using 5-point centred difference formula in row-wise natural ordering.

[25 marks]

(b) Decide the convergence or divergence of Jacobi and Gauss -Seidel iterations for the linear solution $A\bar{x} = \bar{b}$ if $A = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 6 & 4 \\ 4 & 4 & 5 \end{pmatrix}$.

[25 marks]

(c) The linear system of equations $\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}\bar{x} = \bar{b}$ where a is real, can under certain conditions be solved by the iterative method

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}\bar{x}^{(k+1)} = \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}\bar{x}^{(k)} + \omega\bar{b},$$

(i) for which values of a is the method convergent when $\omega = 1$?

(ii) for a =0.5, find the value of ω which minimizes the spectral radius of the matrix $\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}$.

[50 marks]

3. (a) Pertimbangkan masalah nilai sempadan

$$\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 3, 0 < y < 3;$$

$$u(0, y) = 0, u(3, y) = 0,$$

$$\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{3}{4}, \frac{\partial u}{\partial y}\Big|_{y=3} = \frac{5}{4}.$$

Diberi $\Delta x = \Delta y = 1$, *diskretkan persamaan pembezaan di atas menggunakan teknik anggaran beza ketengah lima-titik dalam tertib baris semulajadi.*

[25 markah]

(b) Tentukan penumpuan atau pencapahan lelaran Jacobi dan Gauss-Seidel bagi penyelesaian sistem linear $A\bar{x} = \bar{b}$ *jika* $A = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 6 & 4 \\ 4 & 4 & 5 \end{pmatrix}$.

[25 marks]

(c) *Sistem persamaan linear* $\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}\bar{x} = \bar{b}$ *dimana* a *adalah nyata, boleh di bawah syarat tertentu diselesaikan dengan kaedah lelaran*

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}\bar{x}^{(k+1)} = \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}\bar{x}^{(k)} + \omega\bar{b},$$

(i) *bagi nilai* a *yang manakah kaedah ini akan menumpu untuk* $\omega = 1$?

(ii) *bagi* a *=0.5, dapatkan nilai* ω *yang meminimumkan jejari spektrum matriks* $\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}$.

[50 markah]

4. (a) Given the equations

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1, \\ \frac{x_1}{8} + x_2 + x_3 &= 3, \\ -x_1 + 4x_2 + x_3 &= 7. \end{aligned}$$

(i) Write down the formula for the second order Richardson's method to solve this system.

(ii) What is the rate of convergence, R_∞, of this iterative method in solving this system?

[50 marks]

(b) Prove that the eigenvalues λ of the S.O.R. iteration matrix in solving the system $A\bar{u} = \bar{b}$, are the roots of $\det\{(\lambda+\omega-1)D - \lambda\omega L - \omega U\} = 0$. Here, $A = D - L - U$ with D, L and U being diagonal, lower triangular and upper triangular matrices respectively.

[25 marks]

(c) Prove that the truncation error of the five-point finite difference formula approximating Laplace's equation at the point (x_i, y_j) for a square mesh of side h can be written as

$$\frac{1}{12}h^2\left\{\frac{\partial^4}{\partial x^4}U(\xi, y_j) + \frac{\partial^4}{\partial x^4}U(x_i, \eta)\right\},$$

where $x_i - h < \xi < x_i + h, y_j - h < \eta < y_j + h,$ and it is assumed that the first-, second-, third- and fourth- order partial derivatives of U with respect to x and y are continuous throughout these intervals respectively.

[25 marks]

4. (a) *Diberikan persamaan seperti di bawah*

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1, \\ \frac{x_1}{8} + x_2 + x_3 &= 3, \\ -x_1 + 4x_2 + x_3 &= 7. \end{aligned}$$

(i) *Tuliskan rumus kaedah Richardson peringkat dua untuk menyelesaikan sistem ini.*

(ii) *Apakah kadar penumpuan,* R_∞*, bagi kaedah ini dalam menyelesaikan sistem tersebut?*

[50 markah]

(b) *Buktikan bahawa nilai eigen* λ *bagi matriks lelaran S.O.R. dalam menyelesaikan sistem* $A\bar{u} = \bar{b}$*, ialah punca persamaan* $pen\{(\lambda + \omega - 1)D - \lambda\omega L - \omega U\} = 0$. *Disini* $A = D - L - U$ *dengan* D, L *dan* U *adalah masing-masing matriks pepenjuru, segitiga bawah dan segitiga atas.*

[25 markah]

(c) *Buktikan bahawa ralat pangkasan untuk rumus pembezaan lima-titik untuk menganggarkan persamaan Laplace pada titik* (x_i, y_j) *pada segi empat mesh bersisi h boleh ditulis sebagai*

$$\frac{1}{12}h^2\left\{\frac{\partial^4}{\partial x^4}U(\xi, y_j) + \frac{\partial^4}{\partial x^4}U(x_i, \eta)\right\},$$

dimana $x_i - h < \xi < x_i + h, y_j - h < \eta < y_j + h,$ *dan diandaikan peringkat pertama, kedua, ketiga dan keempat persamaan pembezaan* U *terhadap* x *dan* y *adalah masing-masing selanjar pada semua selang.*

[25 markah]

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