UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2011/2012

Ogos 2012

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) If $X_1, X_2, ..., X_n$ is a random sample from the normal distribution, $N(\mu, 3\sigma^2)$ and $\bar{X}_m$ is defined as

$$\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i \ , \quad m \le n,$$

find the distributions of the following statistics:

(i) $\bar{X}_m + \bar{X}_n$.

(ii) $\dfrac{m(\bar{X}_m - 2\mu)^2}{3\sigma^2}$.

[40 marks]

(b) The random variables U and V have a joint probability density function (pdf)

$$f(u, v) = \frac{1}{4}uv, \quad 0 \le u \le v \le 2.$$

Find

(i) the conditional density function of U given $V = v$.

(ii) the conditional mean of U given $V = 2$.

[40 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from the $Be(\theta)$ distribution, where $0 < \theta < 1$. If $\bar{X}_n$ denotes the sample mean, show that $\bar{X}_n$ converges in probability to θ.

[20 marks]

1. (a) *Jika $X_1, X_2, ..., X_n$ adalah suatu sampel rawak daripada taburan normal, $N(\mu, 3\sigma^2)$ dan $\bar{X}_m$ ditakrifkan sebagai*

$$\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i \ , \quad m \le n,$$

cari taburan untuk statistik berikut:

(i) $\bar{X}_m + \bar{X}_n$.

(ii) $\dfrac{m(\bar{X}_m - 2\mu)^2}{3\sigma^2}$.

[40 markah]

(b) Pembolehubah rawak U dan V mempunyai fungsi ketumpatan kebarangkalian (fkk) tercantum

$$f(u, v) = \frac{1}{4}uv, \quad 0 \le u \le v \le 2.$$

Cari

(i) fungsi ketumpatan bersyarat U diberi $V = v$.

(ii) min bersyarat U diberi $V = 2$.

[40 markah]

(c) Andaikan bahawa $X_1, X_2, ..., X_n$ *adalah suatu sampel rawak daripada taburan* $Be(\theta)$ *, yang mana* $0 < \theta < 1$. *Jika* $\overline{X}_n$ *mewakili min sampel, tunjukkan bahawa* $\overline{X}_n$ *menumpu secara kebarangkalian kepada* θ.

[20 markah]

2. (a) If Y_n denotes the n^{th} ordered statistic of a random sample from the uniform distribution, $U(0, \lambda)$, find the limiting distribution of $Z_n = n(\lambda - Y_n)$.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample of size n with probability mass function (pmf), $f(x;\lambda) = e^{-\lambda}\dfrac{\lambda^x}{x!}$, for $x = 0, 1, 2, \ldots ;\ \lambda > 0$. Find the maximum likelihood estimator (mle) of λ.

[20 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from the normal distribution, $N(0, \theta)$, $\theta > 0$.

(i) Show that $f(x;\theta)$ is an exponential family.

(ii) From (i), find a complete and sufficient statistic.

(iii) Is $\overline{X}$ an uniformly minimum variance of unbiased estimator (UMVUE) of θ? Explain.

[50 marks]

2. (a) Jika Y_n *mewakili statistik tertib ke-n bagi suatu sampel rawak daripada taburan seragam,* $U(0, \lambda)$, *cari taburan penghad untuk* $Z_n = n(\lambda - Y_n)$.

[30 markah]

(b) Biarkan $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak saiz n dengan fungsi jisim kebarangkalian (fjk),* $f(x;\lambda) = e^{-\lambda}\dfrac{\lambda^x}{x!}$, *untuk* $x = 0, 1, 2, \ldots ;\ \lambda > 0$. *Cari penganggar kebolehjadian maksimum (pkm) untuk* λ.

[20 markah]

(c) Andaikan bahawa $X_1, X_2, ..., X_n$ *adalah suatu sampel rawak daripada taburan normal,* $N(0, \theta)$, $\theta > 0$.

(i) Tunjukkan bahawa $f(x;\theta)$ *adalah suatu famili eksponen.*

(ii) Daripada (i), cari suatu statistik cukup dan lengkap.

(iii) Adakah $\overline{X}$ *suatu penganggar saksama bervarians minimum secara seragam (PSVMS) untuk* θ? *Jelaskan.*

[50 markah]

3. (a) Assume that X_1 and X_2 are random variables from the Poisson distribution, $P(\theta)$. Is $X_1 + 2X_2$ a sufficient statistic for θ?

[30 marks]

(b) Let X be a single observation from a distribution with pdf

$$f(x;\lambda) = \frac{2}{\lambda^2}(\lambda - x) I_{(0,\lambda)}(x), \quad \lambda > 0.$$

If $\left(\frac{X}{d}, \frac{X}{c}\right)$ is a confidence interval for λ, find the confidence coefficient, in terms of c and d.

[30 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from the exponential distribution with parameter θ. Based on the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, derive

(i) the 100γ% approximate confidence interval for θ when n is large.

(ii) the 100γ% exact confidence interval for θ when n is small.

[40 marks]

3. (a) Andaikan bahawa X_1 dan X_2 adalah pembolehubah rawak daripada taburan Poisson, $P(\theta)$. Adakah $X_1 + 2X_2$ suatu statistik cukup bagi θ?

[30 markah]

(b) Biarkan X sebagai suatu cerapan tunggal daripada taburan dengan fkk

$$f(x;\lambda) = \frac{2}{\lambda^2}(\lambda - x) I_{(0,\lambda)}(x), \quad \lambda > 0.$$

Jika $\left(\frac{X}{d}, \frac{X}{c}\right)$ adalah suatu selang keyakinan bagi λ, cari pekali keyakinan, dalam sebutan c dan d.

[30 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ sebagai suatu sampel rawak daripada taburan eksponen dengan parameter θ. Berdasarkan min sampel $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, terbitkan

(i) selang keyakinan hampiran 100γ% bagi θ apabila n adalah besar.

(ii) selang keyakinan tepat 100γ% bagi θ apabila n adalah kecil.

[40 markah]

4. (a) Let $X_1, X_2, ..., X_n$ denote a random sample of size n having pdf $f(x;\theta) = \theta^2 x e^{-\theta x} I_{(0,\infty)}(x)$.

(i) Find the uniformly most powerful critical region to test $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

(ii) For testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$, the following test is used: Reject H_0 if and only if $\left|\overline{X} - 2\right| \geq k$. By assuming that n is sufficiently large, find the approximate value of k using the Central Limit Theorem so that $\alpha = 0.05$.

[50 marks]

(b) Assume that $X_1, X_2, ..., X_{20}$ is a random sample of size 20 from a normal distribution, $N(0, \theta)$, where $\theta > 0$. For testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$, the following critical region is used:

$$C = \left\{ (x_1, x_2, \ldots, x_{20}) : \sum_{i=1}^{20} x_i^2 \geq c \right\}.$$

Find the value of c if the size of the critical region C is 0.10.

[30 marks]

(c) Assume that X is a single observation having pdf $f(x;\theta) = (1+\theta)x^{\theta} I_{(0,1)}(x)$, where $\theta > -1$. For testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$, the following test is used: Reject H_0 if and only if $X \geq \frac{1}{2}$. Find the power function of this test.

[20 marks]

4. (a) *Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz n yang mempunyai fkk $f(x;\theta) = \theta^2 x e^{-\theta x} I_{(0,\infty)}(x)$.*

(i) Cari rantau genting paling berkuasa secara seragam untuk menguji $H_0 : \theta = 1$ lawan $H_1 : \theta > 1$.

(ii) Untuk menguji $H_0 : \theta = 1$ lawan $H_1 : \theta \neq 1$, ujian berikut digunakan: Tolak H_0 jika dan hanya jika $\left|\overline{X} - 2\right| \geq k$. Dengan mengandaikan bahawa n adalah besar secara cukup, cari nilai hampiran k dengan menggunakan Teorem Had Memusat supaya $\alpha = 0.05$.

[50 markah]

(b) *Andaikan bahawa $X_1, X_2, ..., X_{20}$ adalah suatu sampel rawak saiz 20 daripada taburan normal, $N(0, \theta)$, yang mana $\theta > 0$. Untuk menguji $H_0 : \theta = 1$ lawan $H_1 : \theta > 1$, rantau genting berikut digunakan:*

$$C = \left\{ (x_1, x_2, \ldots, x_{20}) : \sum_{i=1}^{20} x_i^2 \geq c \right\}.$$

Cari nilai c jika saiz rantau genting C adalah 0.10.

[30 markah]

(c) *Andaikan bahawa X adalah cerapan tunggal yang mempunyai fkk* $f(x;\theta) = (1+\theta)x^{\theta} I_{(0,1)}(x)$*, yang mana* $\theta > -1$*. Untuk menguji* $H_0 : \theta \leq 0$ *lawan* $H_1 : \theta > 0$ *, ujian berikut digunakan: Tolak* H_0 *jika dan hanya jika* $X \geq \frac{1}{2}$*. Cari fungsi kuasa ujian ini.*

[20 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\,e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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