UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2010/2011 Academic Session

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MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TWELVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi DUA BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Let E, F and G be the events in a sample space. State whether each of the following statements is true or false. Justify your answers.

   (a) If E and F are mutually exclusive, then they are independent.
   (b) If E and F are independent, then they are mutually exclusive.
   (c) $P(E) = P(F) = 0.7$, and E and F are mutually exclusive.
   (d) If E is independent of F, and E is independent of G, and $F \cap G = \varnothing$, then G is independent of $F \cup G$.

   [20 marks]

2. An insurance company believes that factories can be divided into two types: those which are fire prone and those which are not. The company's statistics show that a fire-prone factory will have a fire accident at some time within a fixed 1-year period with probability 0.2, whereas this probability is 0.05 for a factory that is not fire prone. Assume that 20% of the factories in an industrial zone is fire prone.

   (a) What is the probability that a factory will have a fire accident within a year of purchasing a fire policy?

   (b) Suppose that a new purchaser has a fire accident within a year of purchasing an insurance policy. What is the probability that the factory is fire prone?

   (c) What is the probability that a factory who purchased a policy will have a fire accident in its second year of policy ownership, given that the factory has had a fire in the first year?

   [25 marks]

1. *Katakan* E, F *dan* G *adalah peristiwa dalam suatu ruang sampel. Nyatakan samada setiap kenyataan berikut adalah benar atau palsu. Tentusahkan jawapan anda.*

   (a) *Jika* E *dan* F *adalah saling eksklusif, maka mereka adalah tak bersandar.*
   (b) *Jika* E *dan* F *adalah tak bersandar, maka mereka adalah saling eksklusif.*
   (c) $P(E) = P(F) = 0.7$, *dan* E *dan* F *adalah saling eksklusif.*
   (d) *Jika* E *adalah tak bersandar dengan* F, *dan* E *adalah tak bersandar dengan* G, *dan* $F \cap G = \varnothing$, *maka* G *adalah tak bersandar dengan* $F \cup G$.

[20 markah]

2. *Sebuah syarikat insuran percaya bahawa kilang boleh dibahagikan kepada dua jenis: yang mudah terbakar dan yang tidak mudah terbakar. Statistik syarikat menunjukkan bahawa sebuah kilang mudah terbakar akan menghadapi kebakaran pada bila-bila masa dalam suatu tempoh tetap satu tahun dengan kebarangkalian 0.2, sedangkan kebarangkalian ini adalah 0.05 bagi sebuah kilang yang tidak mudah terbakar. Andaikan bahawa 20% daripada kilang dalam suatu zon industri adalah mudah terbakar.*

   (a) *Apakah kebarangkalian bahawa sebuah kilang akan menghadapi kebakaran dalam tempoh satu tahun pembelian polisi insuran kebakaran?*

   (b) *Katakan bahawa suatu pembeli baru menghadapi kebakaran dalam tempoh satu tahun pembelian polisi insuran. Apakah kebarangkalian bahawa kilang ini adalah mudah terbakar?*

   (c) *Apakah kebarangkalian bahawa sebuah kilang yang membeli suatu polisi akan menghadapi kebakaran dalam tahun kedua ia memegangi polisi, diketahui bahawa kilang ini telah menghadapi kebakaran dalam tahun pertama?*

[25 markah]

3. A random variable X has a c.d.f.

$$F(x)=\begin{cases} 0 & ; \quad x<0 \\ \frac{1}{6} & ; \quad 0\le x<1 \\ \frac{2}{3} & ; \quad 1\le x<2 \\ \frac{11}{12} & ; \quad 2\le x<3 \\ 1 & ; \quad 3\le x \end{cases}$$

(a) Find the p.d.f. of X. Hence, calculate $P(0.5<X<3)$.

(b) Suppose $X_i;\ i=1,2$ are independent random variables that have a common distribution as X. If $W=X_1+X_2$, find the moment generating function of W. Hence, obtain the p.d.f. of W from this moment generating function.

[25 marks]

4. Suppose that a batch of 100 items contains 6 that are defective and 94 that are not defective. Find $P(X>2)$ for each of the following experiments.

(a) Let X be the number of defective items in a randomly drawn sample of 10 items from the batch.

(b) A sampling with replacement is done until a defective item is obtained and X is the number of trials needed.

(c) A sampling with replacement is performed until the third defective item is obtained and X is the number of trials needed.

[15 marks]

3. *Suatu pemboleh ubah rawak X mempunyai suatu f.k.m.*

$$F(x)=\begin{cases} 0 & ; \quad x<0 \\ \frac{1}{6} & ; \quad 0\le x<1 \\ \frac{2}{3} & ; \quad 1\le x<2 \\ \frac{11}{12} & ; \quad 2\le x<3 \\ 1 & ; \quad 3\le x \end{cases}$$

(a) *Cari f.k.k. bagi X. Dengan itu, kira $P(0.5<X<3)$.*

(b) *Katakan X_i; $i=1,2$ adalah pemboleh ubah rawak tak bersandar yang mempunyai suatu taburan sepunya seperti X. Jika $W=X_1+X_2$, cari fungsi penjana momen bagi W. Dengan itu, dapatkan f.k.k. bagi W dari fungsi penjana momen ini.*

[25 markah]

4. *Katakan bahawa suatu kelompok 100 butir mengandungi 6 yang cacat dan 94 yang tidak cacat. Cari $P(X>2)$ bagi setiap ujikaji berikut.*

(a) *Katakan X ialah bilangan butir cacat dalam suatu sampel 10 butir yang diambil secara rawak dari kelompok ini.*

(b) *Suatu pensampelan dengan pengembalian dijalankan sehingga satu butir cacat diperolehi dan X ialah bilangan cubaan yang diperlukan.*

(c) *Suatu pensampelan dengan pengembalian dijalankan sehingga butir cacat ketiga diperolehi dan X ialah bilangan cubaan yang diperlukan.*

[15 markah]

5. Let X be a binomial random variable with parameters n and p. Find $E\left[\dfrac{1}{X+1}\right]$.

[15 marks]

6. Fifty-two percent of the people who live in Penang are in favor of using public transport. Let N denote the number of people who live in Penang. A random sample of size n is chosen and T denote the number of people in the sample who are in favor of public transport.

(a) State the exact distribution of T.

(b) What is another discrete distribution that can be used to approximate the probability related to T?

(c) Using a continuous distribution approximation, how large n have to be to make the probability that more than 50 percent of a random sample of n people are in favor of public transport is at least 0.95?

[20 marks]

7. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{e^{-y}}{y}; & 0 < x < y, \quad 0 < y < \infty \\ 0 \quad ; & \text{otherwise} \end{cases}$$

(a) Find $E[X]$.

(b) Compute $E[X^3 \mid Y = y]$.

(c) Find $P(X < 2 \mid Y > 2)$.

[30 marks]

5. *Katakan X ialah suatu pemboleh ubah rawak Binomial dengan parameter n dan p. Cari* $E\left[\frac{1}{X+1}\right]$.

[15 markah]

6. *Lima puluh dua peratus orang yang tinggal di Pulau Pinang menyokong penggunaan kenderaan awam. Katakan N adalah orang yang tinggal di Pulau Pinang. Suatu sampel rawak bersaiz n dipilih dan T adalah bilangan orang dalam sampel yang menyokong penggunaan kenderaan awam.*

   (a) *Nyatakan taburan tepat bagi* T.
   (b) *Apakah taburan diskret lain yang boleh digunakan untuk menganggar kebarangkalian berkaitan dengan T ?*
   (c) *Menggunakan suatu penghampiran selanjar, berapa besar n yang perlu diambil supaya kebarangkalian lebih dari 50 peratus dalam suatu sampel rawak n orang menyokong penggunaan kenderaan awam adalah sekurang-kurangnya 0.95?*

[20 markah]

7. *Suatu ketumpatan tercantum X dan Y diberi oleh*

$$f(x,y)=\begin{cases}\dfrac{e^{-y}}{y}; & 0<x<y,\quad 0<y<\infty\\ 0\ ; & \text{di tempat lain.}\end{cases}$$

   (a) *Cari* $E[X]$.
   (b) *Kira* $E[X^3 \mid Y=y]$.
   (c) *Cari* $P(X<2 \mid Y>2)$.

[30 markah]

8. Let $f(x,y)$ denote the joint p.d.f. of two continuous random variables X and Y.

(a) Show that $E(X \mid y) = \mu_X + \rho \dfrac{\sigma_X}{\sigma_Y}(y - \mu_Y)$.

(b) Given

$$f(x,y) = \begin{cases} 1 & ; \quad -x < y < x, \quad 0 < x < 1, \\ 0 & ; \qquad \text{otherwise,} \end{cases}$$

show that $E(Y \mid x)$ is a straight line, whereas that of $E(X \mid y)$ is not a straight line.

[30 marks]

9. (a) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-x(y+1)}; & 0 < x, \quad 0 < y \\ 0 \qquad ; & \text{otherwise} \end{cases}.$$

Find the distribution of $Z = XY$.

(b) If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of $U = X + Y, V = X / Y$. Are U and V independent?

[30 marks]

10. Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with mean 1.

(a) Obtain a bound on $P\left(\sum_{i=1}^{20} X_i > 15\right)$. Compare this value with the value approximated by the central limit theorem.

(b) Assuming another sampling will be performed, use Chebyshev's inequality to find the sample size needed if we want to be 70% certain that the difference between the $\sum X_i$ and its mean is within ± 10.

[30 marks]

8. *Katakan* $f(x,y)$ *adalah f.k.k. tercantum bagi dua pemboleh ubah rawak selanjar* X *dan* Y.

(a) *Tunjukkan bahawa* $E(X \mid y) = \mu_X + \rho \dfrac{\sigma_X}{\sigma_Y}(y - \mu_Y)$.

(b) *Diberi*

$$f(x,y) = \begin{cases} 1 & ; \quad -x < y < x, \quad 0 < x < 1 \\ 0 & ; \quad \textit{di tempat lain,} \end{cases}$$

tunjukkan bahawa $E(Y \mid x)$ *adalah suatu garis lurus, manakala* $E(X \mid y)$ *adalah bukan suatu garis lurus.*

[30 markah]

9. (a) *Suatu fungsi ketumpatan bagi* X *dan* Y *diberi oleh*

$$f(x,y) = \begin{cases} xe^{-x(y+1)}; & 0 < x, \quad 0 < y, \\ 0 & ; \quad \textit{di tempat lain.} \end{cases}$$

Cari taburan bagi $Z = XY$.

(b) *Jika* X *dan* Y *adalah pemboleh ubah rawak seragam yang tertabur secara secaman dan tak bersandar pada selang (0,1), kira fungsi tercantum bagi* $U = X + Y, V = X/Y$. *Adakah* U *dan* V *tak bersandar?*

[30 markah]

10. *Biar* $X_1, \ldots, X_{20}$ *sebagai pemboleh ubah rawak Poisson tak bersandar dengan min 1.*

(a) *Dapatkan batas ke atas* $P\left(\sum_{i=1}^{20} X_i > 15\right)$. *Bandingkan nilai ini dengan nilai yang dihampirkan oleh teorem had memusat.*

(b) *Seandainya suatu lagi pensampelan akan dijalankan, guna ketaksamaan Chebyshev untuk menentukan saiz sampel yang diperlukan jika kita ingin 70% pasti bahawa beza antara* $\sum X_i$ *dan minnya adalah dalam lingkungan* ± 10.

[30 markah]

APPENDIX

DISCRETE DISTRIBUTIONS	
Bernoulli	$f(x) = p^x (1-p)^{1-x}, \quad x = 0,1$ $M(t) = 1-p+pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial	$f(x) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$ $M(t) = (1-p+pe^t)^n$ $\mu = p, \quad \sigma^2 = np(1-p)$
Geometric	$f(x) = (1-p)^x p, \quad x = 0,1,2,\ldots$ $M(t) = \dfrac{p}{1-(1-p)e^t}, \quad t < -\ln(1-p)$ $\mu = \dfrac{1-p}{p}, \quad \sigma^2 = \mu = \dfrac{1-p}{p^2}$
Negative Binomial	$f(x) = \dfrac{(x+r-1)!}{x!(r-1)!} p^r (1-p)^x, \quad x = 0,1,2,\ldots$ $M(t) = \dfrac{p^r}{\left[1-(1-p)e^t\right]^r}, \quad t < \ln(1-p)$ $\mu = \dfrac{r(1-p)}{p}, \quad \sigma^2 = \dfrac{r(1-p)}{p^2}$
Poisson	$f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Hypergeometric	$f(x) = \dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{t}}, \quad x \le r, \quad x \le n_1, \quad r-x \le n_2,$ $\mu = \dfrac{rn_1}{n}, \quad \sigma^2 = \dfrac{rn_1n_2(n-r)}{n^2(n-1)}$

CONTINUOUS DISTRIBUTION	
Uniform	$f(x)=\frac{1}{b-a}, \quad a \le x \le b$ $M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)}, \quad t \ne 0, \quad M(0)=1$ $\mu=\frac{a+b}{2}, \quad \sigma^2=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-x/\theta}, \ 0 \le x \le \infty$ $M(t)=\frac{1}{1-\theta t}, \quad t<1/\theta$ $\mu=\theta, \quad \sigma^2=\theta^2$
Gamma	$f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-x/\theta}, \quad 0 \le x \le \infty$ $M(t)=\frac{1}{(1-\theta t)^{\alpha}}, \quad t<1/\theta$ $\mu=\alpha\theta, \quad \sigma^2=\alpha\theta^2$
Chi Square	$f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2}, \quad 0 \le x \le \infty$ $M(t)=\frac{1}{(1-2t)^{r/2}}, \quad t<\frac{1}{2}$ $\mu=r, \quad \sigma^2=2r$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left[(x-\mu)^2/2\sigma^2\right]}, \quad -\infty<x<\infty$ $M(t)=e^{\mu t+\sigma^2 t^2/2}$ $E(X)=\mu, \quad Var(X)=\sigma^2$
Beta	$f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0<x<1$ $\mu=\frac{\alpha}{\alpha+\beta}, \quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

	FORMULA
1.	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
2.	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad \lvert r \rvert < 1$
3.	$\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = \ a+b\ ^n$
4.	$\sum_{x=0}^{n} \binom{n}{x} \binom{r-n}{r-x} = \binom{n}{r}$
5.	$\sum_{x=0}^{n} \binom{n+k-1}{k} w^k = \ 1-w\ ^{-n}, \quad \lvert w \rvert < 1$
6.	$\Gamma\ \alpha\ = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \Gamma\ \alpha\ = \ \alpha-1\ !$
7.	$B\ \alpha,\beta\ = \int_0^1 x^{\alpha-1}\ 1-x\ ^{\beta-1} dx$
8.	$B\ \alpha,\beta\ = \frac{\Gamma\ \alpha\ \Gamma\ \beta}{\Gamma\ \alpha+\beta}$
9.	Polar coordinates: $y = r\cos\theta$, $z = r\sin\theta$

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