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UNIVERSITI SAINS MALAYSIA

Second Semester Examination  
2010/2011 Academic Session

April/May 2011

**MST 565 – Linear Models**  
***[Model Linear]***

Duration : 3 hours  
*[Masa : 3 jam]*

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Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions:** Answer **all four** [4] questions.

**Arahan:** Jawab **semua empat** [4] soalan.]

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].*

1. (a) Find the generalized inverse of matrix  $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$ .

[7 marks]

- (b) Suppose matrix  $\mathbf{A}$  is  $n \times p$  of rank  $r$  and that  $\mathbf{A}$  is partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

where  $\mathbf{A}_{11}$  is  $r \times r$  of rank  $r$ . Show that the generalized inverse of matrix  $\mathbf{A}$  is given as

$$\mathbf{A}^- = \begin{pmatrix} \mathbf{A}_{11}^- & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

with the three  $\mathbf{O}$  matrices are of appropriate sizes so that  $\mathbf{A}^-$  is  $p \times n$ .

[10 marks]

- (c) If matrix  $\mathbf{A}$  is  $n \times p$  and matrix  $\mathbf{B}$  is  $p \times n$ , then show that  $tr \mathbf{AB} = tr \mathbf{BA}$ .

[8 marks]

2. (a) Show that the columns of the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

are mutually orthogonal. Then normalize the three columns of matrix  $\mathbf{A}$  to find matrix  $\mathbf{C}$  such that  $\mathbf{C}'\mathbf{C} = \mathbf{C}\mathbf{C}' = \mathbf{I}$ .

[8 marks]

- (b) Let the random vector  $\mathbf{v}$  be  $N_4 \boldsymbol{\mu}, \boldsymbol{\Sigma}$ , where

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}$$

If  $\mathbf{v}$  is partitioned as  $\mathbf{v} = (y, x_1, x_2, x_3)'$ , find the conditional distribution for  $y | x_1, x_2, x_3$ .

[10 marks]

1. (a) Cari songsangan teritlak bagi matrik  $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$ .

[7 markah]

(b) Katakan matrik  $\mathbf{A}$  adalah  $n \times p$  berpangkat  $r$  dan  $\mathbf{A}$  berpetakan sebagai

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

di mana  $\mathbf{A}_{11}$  adalah  $r \times r$  berpangkat  $r$ . Tunjukkan bahawa songsangan teritlak bagi matrik  $\mathbf{A}$  diberikan sebagai

$$\mathbf{A}^- = \begin{pmatrix} \mathbf{A}_{11}^- & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

dengan ketiga-tiga matrik  $\mathbf{O}$  dengan saiz yang bersesuaian supaya  $\mathbf{A}^-$  ialah  $p \times n$ .

[10 markah]

(c) Jika matrik  $\mathbf{A}$  ialah  $n \times p$  dan matrik  $\mathbf{B}$  ialah  $p \times n$ , maka tunjukkan bahawa  $tr \mathbf{AB} = tr \mathbf{BA}$ .

[8 markah]

2. (a) Tunjukkan bahawa lajur bagi matrik berikut

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

adalah saling ortogon. Seterusnya piawaikan ketiga-tiga lajur matrik  $\mathbf{A}$  untuk mendapatkan matrik  $\mathbf{C}$  supaya  $\mathbf{C}'\mathbf{C} = \mathbf{C}\mathbf{C}' = \mathbf{I}$ .

[8 markah]

(b) Biar vektor rawak  $\mathbf{v}$  sebagai  $N_4 \mu, \Sigma$ , yang mana

$$\mu = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}.$$

Jika  $\mathbf{v}$  berpetakan sebagai  $\mathbf{v} = y, x_1, x_2, x_3'$ , cari taburan bersyarat bagi  $y | x_1, x_2, x_3$ .

[10 markah]

- (c) Using the same  $\mu$  and  $\Sigma$  in question 2(b), compute the  $\rho_{12}$  and  $\rho_{12:34}$   
 [7 marks]

3. (a) Express the following sums of squares

$$\sum_{i=1}^n y_i^2 = \sum_{i=1}^n (y_i - \bar{y})^2 + n\bar{y}^2$$

as a quadratic form in  $\mathbf{y}$ , where  $\mathbf{y} = (y_1, y_2, \dots, y_n)'$  be a random sample from a population with mean  $\mu$  and variance  $\sigma^2$ .

[10 marks]

- (b) Following question 3(a), and with the normality assumption of the  $y_i$ 's, discuss the distribution and independence properties for the  $\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2}$  and  $\frac{n\bar{y}^2}{\sigma^2}$ .

[8 marks]

- (c) Assuming that  $\mathbf{y}$  is  $N_3(\mu, \Sigma)$ , where

$$\mu = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

find a symmetric matrix  $\mathbf{A}$  such that  $\mathbf{y}'\mathbf{A}\mathbf{y}$  is  $\chi^2_{3, \lambda}$ . Then, determine the non-centrality parameter  $\lambda$ .

[7 marks]

4. (a) In Europe, average rental rate for land with alfalfa is depend on three variables, such as  $x_1$  = average rent paid per acre for all land,  $x_2$  = average number of dairy cows per square mile and  $x_3$  = proportion of farmland in pasture.

Using the data on land rental in Table 1 ,

- (i) Find  $R^2$  and  $R_a^2$ .
- (ii) Solve the general linear hypothesis test for  $H_0 : 2\beta_1 = 2\beta_2 = \beta_3$ .
- (iii) Using an appropriate approach, run a test to identify the usefulness of the second-order terms in prediction of  $y$ . State your conclusion.

[15 marks]

- (c) Menggunakan  $\mu$  dan  $\Sigma$  seperti dalam soalan 2(b), kira  $\rho_{12}$  dan  $\rho_{12.34}$   
[7 markah]

3. (a) Huraikan hasil tambah kuasa dua yang berikut

$$\sum_{i=1}^n y_i^2 = \sum_{i=1}^n (y_i - \bar{y})^2 + n\bar{y}^2$$

sebagai suatu bentuk kuadratik dalam sebutan  $\mathbf{y}$ , yang mana

$\mathbf{y} = y_1, y_2, \dots, y_n$  merupakan sample rawak daripada populasi dengan min  $\mu$  dan varian  $\sigma^2$ .

[10 markah]

- (b) Lanjutan soalan 3(a), dengan andaian normal bagi  $y_i$ , bincangkan taburan

dan ciri-ciri ketakbersandaran bagi  $\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2}$  dan  $\frac{n\bar{y}^2}{\sigma^2}$ .

[8 markah]

- (c) Andaikan  $\mathbf{y}$  ialah  $N_3 \mu, \Sigma$ , yang mana

$$\mu = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

cari matrik simetri  $\mathbf{A}$  supaya  $\mathbf{y}'\mathbf{A}\mathbf{y}$  merupakan  $\chi^2_{3, \lambda}$ . Kemudian, tentukan parameter tak memusat  $\lambda$ .

[7 markah]

4. (a) Di Eropah, purata kadar sewaan bagi tanah alfalfa bergantung kepada tiga pemboleh ubah iaitu  $x_1 =$  purata sewa dibayar per ekar bagi kesemua tanah,  $x_2 =$  purata bilangan lembu tenusu per batu persegi dan  $x_3 =$  nisbah tanah ladang dalam pasturan.

Menggunakan data sewaan tanah dalam Jadual 1:

- (i) Cari  $R^2$  dan  $R_a^2$ .  
 (ii) Selesaikan hipotesis umum linear bagi  $H_0: 2\beta_1 = 2\beta_2 = \beta_3$ .  
 (iii) Menggunakan pendekatan bersesuaian, jalankan ujian untuk mengenalpasti kegunaan terma peringkat kedua dalam menentukan nilai  $y$ . Nyatakan kesimpulan anda.

[15 markah]

- (b) Consider the model  $y_{ij} = \mu + \tau_i + \varepsilon_{ij}; i=1,2 \quad j=1,2,3$ . Given that matrix  $\mathbf{X}$  and vector  $\boldsymbol{\beta}$  as

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix},$$

show that  $\tau_1 - \tau_2$  is testable then find the estimators for  $\boldsymbol{\beta}$ .

[10 marks]

(b) Pertimbangkan model  $y_{ij} = \mu + \tau_i + \varepsilon_{ij}; i=1,2 \quad j=1,2,3$ . Diberikan bahwa matrik  $\mathbf{X}$  dan vektor  $\boldsymbol{\beta}$  sebagai

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix},$$

tunjukkan bahwa  $\tau_1 - \tau_2$  adalah teruji kemudian, cari penganggar bagi  $\boldsymbol{\beta}$ .

[10 markah]

Table 1: Data on land rental

$y$	$x_1$	$x_2$	$x_3$
18.38	15.50	17.25	0.24
20.00	22.29	18.51	0.20
11.50	12.36	11.13	0.12
25.00	31.84	5.54	0.12
52.50	83.90	5.44	0.04
82.50	72.25	20.37	0.05
25.00	27.14	31.20	0.27
30.67	40.41	4.29	0.10
12.00	12.42	8.69	0.41
61.25	69.42	6.63	0.04
60.00	48.46	27.40	0.12

$y$  = average rent paid per acre of land with alfalfa

$x_1$  = average rent paid per acre for all land

$x_2$  = average number of dairy cows per square mile

$x_3$  = proportion of farmland in pasture.

*Jadual 1: Data sewaan tanah*

$y$	$x_1$	$x_2$	$x_3$
18.38	15.50	17.25	0.24
20.00	22.29	18.51	0.20
11.50	12.36	11.13	0.12
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25.00	27.14	31.20	0.27
30.67	40.41	4.29	0.10
12.00	12.42	8.69	0.41
61.25	69.42	6.63	0.04
60.00	48.46	27.40	0.12

$y$  = purata sewa dibayar per ekar bagi tanah dengan alfalfa

$x_1$  = purata sewa dibayar per ekar bagi kesemua tanah

$x_2$  = purata bilangan lembu tenusu per batu persegi

$x_3$  = nisbah tanah ladang dalam pasturan.