UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2010/2011 Academic Session

April/May 2011

MSS 301 – Complex Analysis
[Analisis Kompleks]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of <u>SEVEN</u> pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>TUJUH</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

<u>Instructions</u>: Answer **<u>all twelve</u>** [12] questions.

*[**<u>Arahan</u>:** Jawab **<u>semua dua belas</u>** [12] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Define a complex number and the operations of addition and multiplication of complex numbers. Show that every complex number can be expressed in its Cartesian form $z = a + ib$, where i is the unit imaginary number.

[20 marks]

2. Solve each equation and express the solution in its Cartesian form :

(a) $z^2 = 3 - 4i$

(b) $\sin z = 2i$

[20 marks]

3. If $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, show that the Cauchy-Riemann equations hold at $z_0 = x_0 + iy_0$:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad ; \quad u_y(x_0, y_0) = -v_x(x_0, y_0) .$$

Deduce that $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$.

[20 marks]

4. Determine where $f(z) = f(x+iy) = x^3 + 9xy^2 - 48x - i(9x^2y + y^3 - 1)$ is differentiable, and find its derivative. Where is f analytic?

[20 marks]

5. (a) Show that $f(z) = e^z$ is periodic with period $2\pi i$. Hence show that f is one-to-one in any open disk of radius π.

(b) Find a branch of $\log(z + 2i)$ satisfying $\log(2i) = \ell n 2 + i\frac{5\pi}{2}$.

(c) Show that $w = \sin z$ is an unbounded analytic (entire) function in the complex plane.

[25 marks]

6. Let γ be a curve in the complex plane, and f be a continuous function over γ. Define $\int_\gamma f(z)\,dz$.

[10 marks]

1. *Takrifkan nombor kompleks, dan operasi-operasi hasil tambah serta hasil darab nombor-nombor kompleks. Tunjukkan setiap nombor kompleks dapat diungkapkan dalam bentuk Cartesan* $z = a + ib$, *dengan* i *sebagai nombor unit khayalan.*

   [20 markah]

2. *Selesaikan setiap persamaan berikut dengan meninggalkan jawapan dalam bentuk Cartesan:*

   (a) $z^2 = 3 - 4i$

   (b) $\sin z = 2i$

   [20 markah]

3. *Jika* $f(z) = u(x, y) + iv(x, y)$ *terbezakan pada* z_0, *tunjukkan bahawa persamaan Cauchy-Riemann dipenuhi pada titik* $z_0 = x_0 + iy_0$:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad ; \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

   Deduksikan bahawa $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$.

   [20 markah]

4. *Tentukan di mana* $f(z) = f(x + iy) = x^3 + 9xy^2 - 48x - i(9x^2y + y^3 - 1)$ *terbezakan, dan dapatkan terbitannya. Di manakah fungsi* f *analisis?*

   [20 markah]

5. (a) *Tunjukkan* $f(z) = e^z$ *adalah berkala dengan kala* $2\pi i$. *Justeru tunjukkan* f *bersifat satu-dengan-satu pada setiap cakera terbuka dengan jejari* π.

   (b) *Dapatkan cabang* $\log(z + 2i)$ *yang memenuhi* $\log(2i) = \ell n 2 + i\frac{5\pi}{2}$.

   (c) *Tunjukkan* $w = \sin z$ *merupakan fungsi analisis (seluruh) yang tak terbatas pada satah kompleks.*

   [25 markah]

6. *Andaikan* γ *suatu lengkung pada satah kompleks, dan* f *adalah selanjar pada* γ.

   Takrifkan $\int_{\gamma} f(z)\, dz$.

   [10 markah]

7. Evaluate the following integrals over the given positively oriented simple closed contour :

(a) $$\oint_{|z|=2} \frac{ze^{-z}}{(2z-1)(z+1)}\,dz$$

(b) $$\oint_{|z|=4} \frac{\sin z}{(z-\pi)^4}\,dz$$

[20 marks]

8. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta.$

[15 marks]

9. Find three Laurent series expansion for the function

$$f(z) = \frac{3z}{z^2-z-2}$$

in powers of z.

[25 marks]

10. (a) With $z = x+iy,$ establish the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad ; \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(b) Deduce that f is analytic in a domain D if and only if $f_{\overline{z}} = 0$.

(c) A function $f(z) = u(x,y) + iv(x,y)$, $z = x+iy,$ is harmonic in a domain D if both u and v are harmonic in D. Explain why an analytic function f is harmonic. Is a harmonic function necessarily analytic?

(d) Show that a twice-continuously differentiable function f is harmonic if and only if $f_{z\overline{z}} = 0.$

[35 marks]

7. *Nilaikan setiap kamiran berikut pada kontur tertutup ringkas berarah positif:*

(a) $\displaystyle\oint_{|z|=2} \frac{ze^{-z}}{(2z-1)(z+1)}\,dz$

(b) $\displaystyle\oint_{|z|=4} \frac{\sin z}{(z-\pi)^4}\,dz$

[20 markah]

8. *Nilaikan* $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta.$

[15 markah]

9. *Dapatkan tiga perkembangan siri Laurent fungsi*

$$f(z) = \frac{3z}{z^2-z-2}$$

dalam kuasa z.

[25 markah]

10. (a) *Dengan* $z = x+iy$, *tunjukkan penjelmaan*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad ; \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(b) *Deduksikan* f *analisis pada domain* D *jika dan hanya jika* $f_{\overline{z}} = 0$.

(c) *Fungsi* $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, *adalah harmonik pada domain* D *seandainya kedua-dua* u *dan* v *adalah harmonic pada* D. *Jelaskan mengapa fungsi analisis* f *adalah harmonik. Adakah semestinya fungsi harmonik juga analisis?*

(d) *Tunjukkan fungsi terbezakan berperingkat dua yang selanjar* f *adalah harmonik jika dan hanya jika* $f_{z\overline{z}} = 0$.

[35 markah]

11. Let f be an entire function satisfying $|f(z)| \le 2 + e|z|^3$. Show that

$$|f(z)| \le 2 + e(|z_0| + r)^3$$

on every circle $|z - z_0| = r$. Deduce that $f^{(4)}(z) = 0$ for all z, and hence f has the form

$$f(z) = az^3 + bz^2 + cz + d,$$

a,b,c and d are complex constants.

[20 marks]

12. Let P be a polynomial of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

(a) Show that $Q(z) = z^n P(1/z)$ is also an $n-th$ degree polynomial satisfying $Q(0) = 1$.

(b) Show that $\max_{|z|=1} |Q(z)| = \max_{|w|=1} |P(w)|$.

(c) Deduce that $\max_{|z|=1} |P(z)| \ge 1$.

[20 marks]

11. *Andaikan* f *fungsi seluruh yang memenuhi* $|f(z)| \le 2+e|z|^3$. *Tunjukkan*

$$|f(z)| \le (2+e)(|z_0|+r)^3$$

pada setiap bulatan $|z-z_0|=r$. *Deduksikan* $f^{(4)}(z)=0$ *pada setiap* z, *dan justeru* f *berbentuk*

$$f(z)=az^3+bz^2+cz+d,$$

dengan a,b,c dan d pemalar kompleks.

[20 markah]

12. *Andaikan* P *suatu polynomial berbentuk*

$$P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0.$$

(a) *Tunjukkan* $Q(z)=z^nP(1/z)$ *juga merupakan polynomial berdarjah* n *yang memenuhi* $Q(0)=1$.

(b) *Tunjukkan* $\operatorname{maks}_{|z|=1}|Q(z)| = \operatorname{maks}_{|w|=1}|P(w)|$.

(c) *Deduksikan* $\operatorname{maks}_{|z|=1}|P(z)| \ge 1$.

[20 markah]

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