UNIVERSITI SAINS MALAYSIA

Second Semester Examination 2010/2011
Academic Session

April/May 2011

MAT 202 Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **FIVE** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LIMA** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all six** [6] questions.

*[Arahan: Jawab **semua enam** [6] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1 (a) Give the definition of a finite set.
Give the definition of a countable set.

(b) Give an example of a set which is infinite and countable.
Give an example of a set which is infinite and uncountable.

(c) Suppose that S is a nonempty set.
(i) If $f : S \to \mathbb{N}$ is one to one, show that S is countable.
(ii) If $f : \mathbb{N} \to S$ is one to one, is S countable? If it is true, prove it. If not, give a counterexample.

[25 marks]

2 The Archimedean Principle says that for any two real numbers a and b, with $a > 0$, there exists a positive integer n such that $na > b$.

(a) Show that for any positive number a, there exists a positive integer n such that $\frac{1}{n} < a$.

(b) If $x \in \mathbb{R}$ and $0 < \epsilon < x$, show that the set

$$S = \left\{ n \in \mathbb{N} : x - \epsilon < \frac{1}{n} < x + \epsilon \right\}$$

is a finite set.

(c) Let x be a limit point of a subset A of $\mathbb{R}$. By using the concept of limit point and the Archimedean Principle, show that there exists a sequence $\{x_n\}$ of A such that $\{x_n\}$ converges to x.

[25 marks]

3 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

(a) By using the $\epsilon - \delta$ concept, show that f has a limit at $x = 0$.

(b) Use the sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

[25 marks]

1 (a) *Beri takrif untuk set terhingga.*
Beri takrif untuk set terbilangkan.

(b) *Beri satu contoh set yang tak terhingga dan terbilangkan.*
Beri satu contoh set yang tak terhingga dan tak terbilangkan.

(c) *Andaikan S set tak kosong.*
(i) Jika $f : S \to \mathbb{N}$ ialah satu ke satu, tunjukkan bahawa S adalah terbilangkan.
(ii) Jika $f : \mathbb{N} \to S$ ialah satu ke satu, adakah S terbilangkan? Jika ia benar, buktikan. Jika tidak, beri satu contoh lawan.

[25 markah]

2 *Prinsip Archimedes mengatakan untuk sebarang dua nombor nyata a dan b, dengan $a > 0$, wujud integer positif n sedemikian $na > b$.*

(a) *Tunjukkan bahawa untuk sebarang nombor positif a, wujud integer positif n supaya $\frac{1}{n} < a$.*

(b) *Jika $x \in \mathbb{R}$ dan $0 < \epsilon < x$, tunjukkan bahawa set*

$$S = \left\{ n \in \mathbb{N} : x - \epsilon < \frac{1}{n} < x + \epsilon \right\}$$

ialah set terhingga.

(c) *Andaikan x titik had untuk subset A kepada $\mathbb{R}$. Dengan menggunakan konsep titik had dan prinsip Archimedes, tunjukkan bahawa wujud jujukan $\{x_n\}$ dalam A sedemikian $\{x_n\}$ menumpu ke x.*

[25 markah]

3 *Andaikan $f : \mathbb{R} \to \mathbb{R}$ ditakrif sebagai* $f(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

(a) *Dengan menggunakan konsep $\epsilon - \delta$, tunjukkan bahawa f mempunyai had pada $x = 0$.*

(b) *Gunakan hujahan berjujukan untuk menunjukkan bahawa jika $c \neq 0$, maka f tidak mempunyai had pada c.*

[25 markah]

4 (a) Give the definition of uniform continuity for a function $f : \mathbb{R} \to \mathbb{R}$.

(b) For $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, if there exists a positive constant K such that $|f(x) - f(y)| \leq K|x - y|$ for all x and y in A, then show that f is uniformly continuous on A.

(c) Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is uniformly continuous on $[0, 1]$ but not uniformly continuous on $\mathbb{R}$.

[25 marks]

5 Consider $f_n(x) = \frac{x^n sin(nx+n)}{n^2(1+x^n)}$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that the sequence $\{f_n\}$ converges uniformly on $[0, \infty)$ and find the limit function.

(b) Does the series $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly on $[0, \infty)$? Give your reason.

(c) Based on the results in (a) and (b), can we conclude that uniform convergence of a sequence $\{f_n\}$ implies uniform convergence of the series $\sum_{n=1}^{\infty} f_n(x)$? If no, give a counterexample.

[25 marks]

6 (a) State the Bolzano-Weierstrass Theorem for sequences in $\mathbb{R}$.

(b) Suppose that A is a closed bounded subset of $\mathbb{R}$ and let x be a limit point of A. Use part (a) to show that there exists a sequence $\{x_n\}$ of A that converges to x.

(c) Determine whether each of the following sets is open, closed or neither. Justify your answer.
(i) $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$
(ii) $\left\{x : f(x) = \frac{1}{1+x^2}\right\}$
(iii) $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

[25 marks]

4 (a) Beri takrif keselanjaran secara seragam untuk suatu fungsi $f : \mathbb{R} \to \mathbb{R}$.

(b) Untuk $f : \mathbb{R} \to \mathbb{R}$, *jika wujud satu pemalar positif* K *supaya* $|f(x) - f(y)| \leq K|x - y|$ *untuk semua* x *dan* y *dalam* $[0, 1]$, *maka tunjukkan bahawa* f *selanjar secara seragam pada* $[0, 1]$.

(c) Tunjukkan bahawa $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ *adalah selanjar secara seragam pada* $[0, 1]$ *tetapi tak selanjar secara seragam pada* $\mathbb{R}$.

[25 markah]

5 Pertimbangkan $f_n(x) = \frac{x^n sin(nx+n)}{n^2(1+x^n)}$ *untuk* $x \in \mathbb{R}$ *da* $n \in \mathbb{N}$.

(a) Tunjukkan bahawa jujukan $\{f_n\}$ *menumpu secara seragam pada* $[0, \infty)$ *dan cari fungsi had tersebut.*

(b) Adakah siri $\sum_{n=1}^{\infty} f_n(x)$ *menumpu secara seragam pada* $[0, \infty)$*? Beri alasan anda.*

(c) Berdasarkan kepada keputusan dalam (a) dan (b), bolehkah kita menyimpulkan bahawa penumpuan secara seragam jujukan $\{f_n\}$ *akan mengimplikasikan penumpuan secara seragam siri* $\sum_{n=1}^{\infty} f_n(x)$*? Jika tidak, beri satu contoh lawan.*

[25 markah]

6 (a) Nyatakan Teorem Bolzano-Weierstrass untuk jujukan dalam $\mathbb{R}$.

(b) Andaikant A *subset tertutup dan terbatas kepada* $\mathbb{R}$ *dan* x *titik had untuk* A. *Gunakan bahagian (a) untuk menunjukkan bahawa wujud satu jujukan* $\{x_n\}$ *dalam* A *yang menumpu ke* x.

(c) Tentukan sama ada setiap satu daripada set-set berikut adalah terbuka, tertutup atau bukan. Berikan alasan untuk jawapan anda.
(i) $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$
(ii) $\left\{x : f(x) = \frac{1}{1+x^2}\right\}$
(iii) $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

[25 markah]

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