UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

# MAT 363 – Statistical Inference
# *[Pentaabiran Statistik]*

Duration : 3 hours
*[Masa : 3 jam]*

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions**: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].*

1. (a) Assume that $X_1, X_2, \ldots, X_n$ are observations of a random sample from the distribution with probability density function (pdf)

$$f(x) = \alpha x^{\alpha-1},\ 0 < x < 1;\ \alpha > 0.$$

If $Y_1 < Y_2 < \ldots < Y_n$ represent the corresponding order statistics, show that $Y_1/Y_n$ and $Y_n$ are stochastically independent.

[30 marks]

(b) Let $X$ be a discrete random variable with probability mass function (pmf)

$$f(x) = \frac{1}{8}\binom{3}{x},\quad x = 0, 1, 2, 3.$$

(i) Find the moment generating function (mgf) of $X$.

(ii) Use the answer in (i) to find $E(X)$ and $\mathrm{Var}(X)$.

[30 marks]

(c) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variables, each having a normal distribution with mean $\mu$ and variance $\sigma^2$. Define the random variables

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}{n-1}.$$

By using the appropriate theorem, show that $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ has a Student-$t$ distribution with $n - 1$ degrees of freedom.

[20 marks]

(d) Assume that both $X$ and $Y$ are positive discrete random variables with joint pmf

$$f_{X,Y}(x, y) = \frac{4!}{x!\,y!\,(4-x-y)!}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{3}\right)^{y}\left(\frac{1}{6}\right)^{4-x-y},\quad 0 \le x + y \le 4.$$

(i) Find the marginal pmf of $X$.

(ii) What is the distribution of $X$?

[20 marks]

1. (a) *Andaikan* $X_1, X_2, ..., X_n$ *adalah cerapan-cerapan untuk suatu sampel rawak daripada taburan dengan fungsi ketumpatan kebarangkalian (fkk)*

$$f(x) = \alpha x^{\alpha-1},\ 0 < x < 1;\ \alpha > 0.$$

*Jika* $Y_1 < Y_2 < ... < Y_n$ *mewakili statistik tertib yang sepadan, tunjukkan bahawa* $Y_1/Y_n$ *dan* $Y_n$ *adalah tak bersandar secara stokastik.*

*[30 markah]*

(b) *Biarkan X sebagai pembolehubah rawak diskrit dengan fungsi jisim kebarangkalian (fjk)*

$$f(x) = \frac{1}{8}\binom{3}{x},\ x = 0, 1, 2, 3.$$

(i) *Cari fungsi penjana momen (fpm) bagi X.*

(ii) *Gunakan jawapan dalam (i) untuk mencari E(X) dan Var(X).*

*[30 markah]*

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai pembolehubah rawak yang bertaburan secaman dan tak bersandar, setiap satu mempunyai taburan normal dengan min* $\mu$ *dan varians* $\sigma^2$. *Takrifkan pembolehubah rawak*

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad dan \quad S^2 = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}{n-1}.$$

*Dengan menggunakan teorem yang bersesuaian, tunjukkan bahawa* $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ *mempunyai taburan Student-t dengan darjah kebebasan* $n - 1$.

*[20 markah]*

(d) *Andaikan bahawa kedua-dua X dan Y adalah pembolehubah-pembolehubah rawak diskrit positif dengan fjk tercantum*

$$f_{X,Y}(x, y) = \frac{4!}{x!\,y!\,(4-x-y)!}\left(\frac{1}{2}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{6}\right)^{4-x-y},\ 0 \le x + y \le 4.$$

(i) *Cari fjk sut bagi X.*

(ii) *Apakah taburan bagi X?*

*[20 markah]*

2. (a) Let the random variable $Y_n$ have a bin($n$, $p$) distribution. Prove that $(Y_n/n)(1-Y_n/n)$ converges in probability to $p(1-p)$.
Hint : If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_nY_n \xrightarrow{P} XY$.

[30 marks]

(b) Let $Y_n$ represent the largest observation of a random sample from a continuous distribution that has cumulative distribution function (cdf) $F(x)$ and pdf $f(x) = F'(x)$. Find the limiting distribution of $Z_n = n[1-F(Y_n)]$.

Hint : If $\lim_{n\to\infty} \psi(n) = 0$, then for every $a$, $\lim_{n\to\infty}\left[1+\frac{a}{n}+\frac{\psi(n)}{n}\right]^n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

[40 marks]

(c) Assume that $X_1, X_2, ..., X_n$ are independently and identically distributed with pdf $f(x;\theta) = \frac{2x}{\theta^2}$, $0 < x \le \theta$, zero elsewhere. Find

(i) the maximum likelihood estimator $\hat{\theta}$ for $\theta$.

(ii) the constant $c$ so that $E(c\hat{\theta}) = \theta$.

[30 marks]

3. (a) Given $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$, zero elsewhere, with $\theta > 0$.

(i) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\theta$.

(ii) Find $E(Y_n)$, where $Y_n$ is the largest observation of a random sample of size $n$ from this distribution.

(iii) From (ii), derive an unbiased estimator of $\theta$.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ represent a random sample from a distribution with pdf $f(x;\theta) = \theta e^{-\theta x}$, $0 < x < \infty$, zero elsewhere, and $\theta > 0$.

(i) Show that $Y = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic.

(ii) Prove that $\frac{n-1}{Y}$ is the uniformly minimum variance of unbiased estimator of $\theta$.

[30 marks]

2. (a) *Biarkan pembolehubah rawak $Y_n$ mempunyai taburan bin(n, p). Buktikan bahawa $\left(Y_n/n\right)\left(1-Y_n/n\right)$ menumpu secara kebarangkalian kepada $p(1-p)$.*

*Petua : Jika $X_n \xrightarrow{P} X$ dan $Y_n \xrightarrow{P} Y$, maka $X_nY_n \xrightarrow{P} XY$.*

[30 markah]

(b) *Biarkan $Y_n$ mewakili cerapan terbesar suatu sampel rawak daripada taburan selanjar yang mempunyai fungsi taburan longgokan (ftl) F(x) dan fkk f(x) = $F'(x)$. Cari taburan penghad untuk $Z_n = n\left[1-F(Y_n)\right]$.*

*Petua : Jika $\underset{n\to\infty}{\text{had}}\,\psi(n)=0$, maka untuk setiap a,*

$$\underset{n\to\infty}{\text{had}}\left[1+\frac{a}{n}+\frac{\psi(n)}{n}\right]^n = \underset{n\to\infty}{\text{had}}\left(1+\frac{a}{n}\right)^n = e^a.$$

[40 markah]

(c) *Andaikan bahawa $X_1, X_2, ..., X_n$ adalah bertaburan secaman dan tak bersandar dengan fkk $f(x;\theta)=\frac{2x}{\theta^2}$, $0 < x \le \theta$, sifar di tempat lain. Cari*

(i) *penganggar kebolehjadian maksimum $\hat{\theta}$ untuk $\theta$.*

(ii) *pemalar c supaya $E\left(c\hat{\theta}\right)=\theta$.*

[30 markah]

3. (a) *Diberi $f(x;\theta)=\frac{1}{\theta}$, $0 < x < \theta$, sifar di tempat lain, dengan $\theta > 0$.*

(i) *Cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama $\theta$.*

(ii) *Cari $E(Y_n)$, yang mana $Y_n$ adalah cerapan terbesar untuk suatu sampel rawak dengan saiz n daripada taburan ini.*

(iii) *Daripada (ii), terbitkan suatu penganggar saksama untuk $\theta$.*

[30 markah]

(b) *Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak daripada taburan dengan fkk $f(x;\theta)=\theta e^{-\theta x}$, $0 < x < \infty$, sifar di tempat lain, dan $\theta > 0$.*

(i) *Tunjukkan bahawa $Y=\sum_{i=1}^{n} X_i$ adalah statistik cukup dan lengkap.*

(ii) *Buktikan bahawa $\frac{n-1}{Y}$ adalah penganggar saksama bervarians minimum secara seragam untuk $\theta$.*

[30 markah]

(c) For a random sample from a certain population, the distribution of the sample range, $R$, is given by

$$f(r) = \begin{cases} \frac{2}{\theta^2}\ \theta - r & , \text{ for } 0 < r < \theta \\ 0 & , \text{ elsewhere} \end{cases}.$$

Find $c$ (in terms of $\alpha$) so that $R < \theta < cR$ is a $100(1 - \alpha)\%$ confidence interval for $\theta$.

[20 marks]

(d) Let $X_i$, for $i = 1, 2, \ldots, n$, be a continuous random variable having a $U(0, 2\theta)$ distribution. Is $\frac{Y_n}{\theta}$ a pivotal quantity, where $Y_n$ is the largest order statistic from a sample of size $n$? Show your solution.

[20 marks]

4. (a) Let $X_1, X_2, \ldots, X_8$ represent a random sample of size $n = 8$ from a Poisson distribution having mean $\mu$. Reject the null hypothesis $H_0 : \mu = 0.5$ and accept the alternative hypothesis $H_1 : \mu > 0.5$ if $\sum_{i=1}^{8} X_i \geq 8$.

(i) Find the power function $\pi(\mu)$ of the test.

(ii) Determine $\pi(0.75)$.

[40 marks]

(b) Let $X_1, X_2, \ldots, X_n$ represent a random sample from a normal, $N(\theta, 1)$ distribution. Find the likelihood ratio test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

[30 marks]

(c) Let $X$ have the pmf $f\ x;\theta\ = \theta^x\ 1-\theta\ ^{1-x}$, $x = 0, 1$, zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ vs. $H_1 : \theta < \frac{1}{2}$ by taking a random sample $X_1, X_2, \ldots, X_5$ of size 5. Find a uniformly most powerful test to test these hypotheses.

[30 marks]

(c) *Untuk suatu sampel rawak daripada populasi tertentu, taburan untuk julat sampel, R, diberi oleh*

$$f(r)=\begin{cases}\dfrac{2}{\theta^2}\;\theta-r & , \ \textit{untuk}\ 0<r<\theta \\ 0 & , \ \textit{di tempat lain}\end{cases}.$$

*Cari c (dalam sebutan* $\alpha$*) supaya* $R < \theta < cR$ *ialah selang keyakinan* $100(1-\alpha)\%$ *untuk* $\theta$.

*[20 markah]*

(d) *Biarkan* $X_i$, *untuk* $i = 1, 2, \ldots, n$, *sebagai pembolehubah rawak selanjar dengan taburan* $U(0, 2\theta)$. *Adakah* $\dfrac{Y_n}{\theta}$ *suatu kuantiti pangsian, yang mana* $Y_n$ *ialah statistik tertib terbesar daripada sampel dengan saiz n? Tunjukkan penyelesaian anda.*

*[20 markah]*

4\. (a) *Biarkan* $X_1, X_2, \ldots, X_8$ *mewakili suatu sampel rawak dengan saiz n = 8 daripada taburan Poisson yang mempunyai min* $\mu$. *Tolak hipotesis nol* $H_0 : \mu = 0.5$ *dan terima hipotesis alternatif* $H_1 : \mu > 0.5$ *jika* $\sum_{i=1}^{8} X_i \geq 8$.

(i) *Cari fungsi kuasa* $\pi(\mu)$ *untuk ujian ini.*

(ii) *Cari* $\pi(0.75)$.

*[40 markah]*

(b) *Biarkan* $X_1, X_2, \ldots, X_n$ *mewakili suatu sampel rawak daripada taburan normal,* $N(\theta, 1)$. *Cari ujian nisbah kebolehjadian untuk menguji* $H_0 : \theta = \theta_0$ *lawan* $H_1 : \theta \neq \theta_0$.

*[30 markah]*

(c) *Biarkan X mempunyai fjk* $f\ \ x;\theta\ \ = \theta^x\ \ 1-\theta\ \ ^{1-x}$, $x = 0, 1$, *sifar di tempat lain. Kita menguji* $H_0 : \theta = \dfrac{1}{2}$ *lawan* $H_1 : \theta < \dfrac{1}{2}$ *dengan mengambil suatu sampel rawak* $X_1, X_2, \ldots, X_5$ *dengan saiz 5. Cari ujian paling berkuasa secara seragam untuk menguji hipotesis-hipotesis ini.*

*[30 markah]*

## APPENDIX / *LAMPIRAN*

| Taburan | Fungsi Ketumpatan | Min | Varians | Fungsi Penjana Momen |
|---|---|---|---|---|
| Seragam Diskrit | $f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$ | $\frac{N+1}{2}$ | $\frac{N^2-1}{12}$ | $\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$ |
| Bernoulli | $f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$ | $p$ | $pq$ | $q+pe^t$ |
| Binomial | $f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$ | $np$ | $npq$ | $\left(q+pe^t\right)^n$ |
| Geometri | $f(x)=pq^x I_{\{0,1,\cdots\}}(x)$ | $\frac{q}{p}$ | $\frac{q}{p^2}$ | $\frac{p}{1-qe^t},\ qe^t<1$ |
| Poisson | $f(x)=e^{-\lambda}\,\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$ | $\lambda$ | $\lambda$ | $exp\{\lambda(e^t-1)\}$ |
| Seragam | $f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$ |
| Normal | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$ | $\mu$ | $\sigma^2$ | $exp\{\mu t+(\sigma t)^2/2\}$ |
| Eksponen | $f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{\lambda}{\lambda-t},\ t<\lambda$ |
| Gama | $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}\, x^{\alpha-1}\, I_{(0,\infty)}(x)$ | $\frac{\alpha}{\lambda}$ | $\frac{\alpha}{\lambda^2}$ | $\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$ |
| Khi Kuasa Dua | $f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}\, x^{(r/2)-1}\, I_{(0,\infty)}(x)$ | $r$ | $2r$ | $\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$ |
| Beta | $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ | |
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