UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MAT 514 – Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of ELEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEBELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Discuss briefly on the technique that is used in solving the engineering problems in mathematical modelling.

   (b) What is the concept of the boundary layer? List down the two-dimensional boundary-layer approximations.

   (c) Consider a steady flow along a semi-infinite two-dimensional surface of small curvature with a free-stream velocity u_∞. Let x be measured along the surface and y normal to the surface. Cut out an infinitesimal stationary control volume of unit depth within the boundary layer and consider the external forces acting on this control volume in the x direction and the momentum fluxes crossing the control surface as shown in Figure 1.

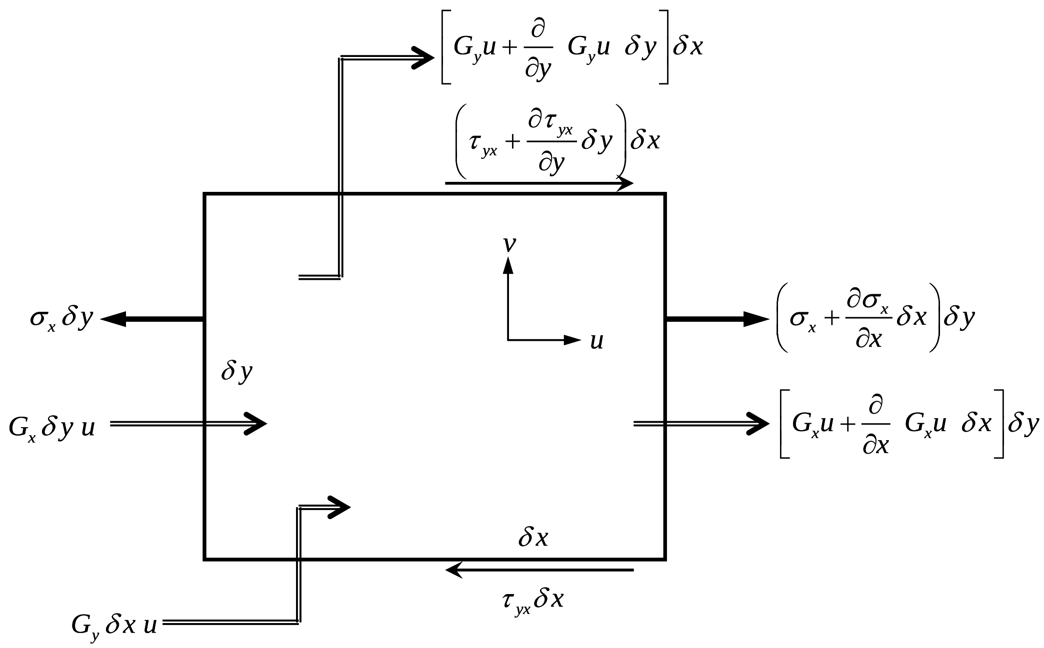


Figure 1: Control volume, momentum fluxes and external forces for development of the steady-flow momentum differential equation of the boundary layer.

Based on Figure 1 and applying the momentum theorem, the boundary-layer approximations in 1(b) and the two-dimensional boundary layer continuity equation $\partial u/\partial x + \partial v/\partial y = 0$, show that the momentum equation of the boundary layer is given by

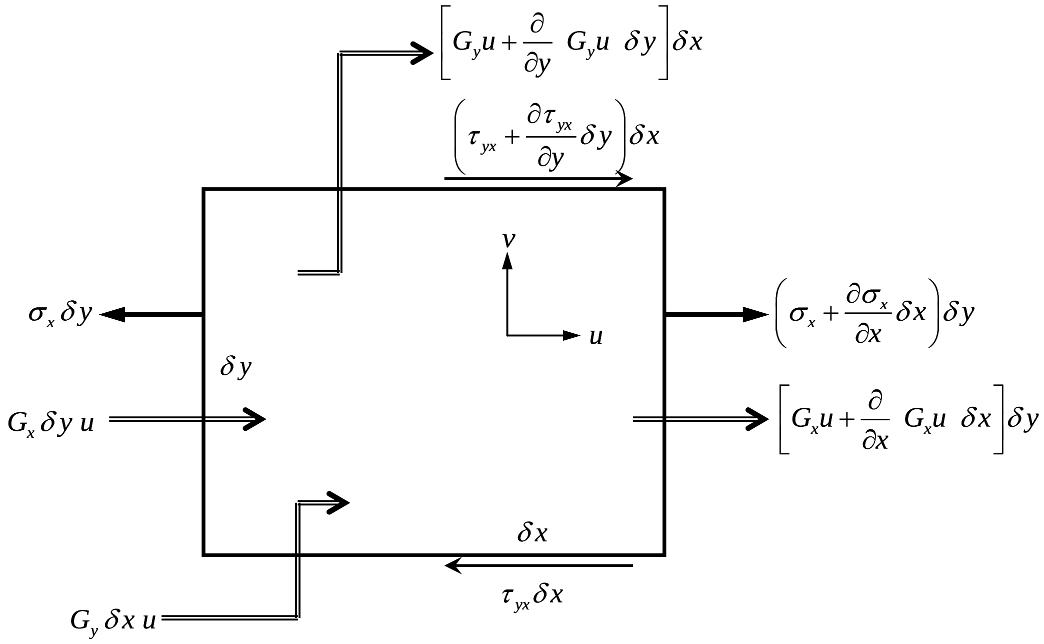
$$G_x \frac{\partial u}{\partial x} + G_y \frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right). \qquad (1)$$

[100 marks]

1. (a) *Bincangkan secara ringkas teknik yang digunakan untuk menyelesaikan masalah kejuruteraan dalam pemodelan matematik.*

   (b) *Apakah konsep lapisan sempadan? Senaraikan penghampiran-penghampiran lapisan sempadan dua dimensi.*

   (c) *Pertimbangkan suatu aliran mantap terhadap permukaan kelengkungan kecil separuh tak terhingga dengan halaju strim bebas u_∞. x diukur di sepanjang permukaan dan y serenjang terhadap permukaan. Satu unit kedalaman unsur isipadu kawalan pegun dalam lapisan sempadan dikeluarkan dan daya-daya luar yang bertindak ke atas unsur isipadu kawalan ini dalam arah x serta fluks-fluks momentum merentasi permukaan kawalan seperti yang ditunjukkan dalam Rajah 1 dipertimbangkan.*



***Rajah 1**: Isipadu kawalan, fluks-fluks momentum dan daya-daya luar untuk penerbitan persamaan pembezaan momentum bagi aliran lapisan sempadan yang mantap.*

Berdasarkan Rajah 1 dan aplikasikan teorem momentum, penghampiran-penghampiran lapisan sempadan dalam 1(b) serta persamaan keselanjaran lapisan sempadan dua dimensi $\partial u/\partial x + \partial v/\partial y = 0$, tunjukkan bahawa persamaan momentum lapisan sempadan diberikan oleh

$$G_x \frac{\partial u}{\partial x} + G_y \frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right). \tag{1}$$

[100 markah]

2. Consider a steady laminar flow of a viscous fluid of temperature T inside a circular tube of radius r_s where x is measured along the length of the tube and r is the radial axis which is normal to x. Let the fluid enter with a uniform velocity over the flow cross section. The momentum equation for axisymmetric flow in this circular tube in $x-$ direction is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u}{\partial r}\right)-\frac{dP}{dx}=0, \qquad (2)$$

where the pressure P is independent of r and the dynamic viscosity of the fluid μ is constant.

(a) Derive the following fully developed velocity profile

$$u=\frac{r_s^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1-\frac{r^2}{r_s^2}\right), \qquad (3)$$

by using the momentum equation (2) subject to the boundary conditions

$$\begin{aligned}\frac{\partial u}{\partial r}&=0 \quad \text{at} \quad r=0,\\ u&=0 \quad \text{at} \quad r=r_s.\end{aligned} \qquad (4)$$

(b) Assuming that the energy equation with constant heat rate for axisymmetric flow in a circular tube is

$$u\left(\frac{dT_m}{dx}\right)=\alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \qquad (5)$$

where T_m and α are the mass-averaged temperature and the constant thermal diffusivity of the fluid, respectively. Derive the fully developed temperature profile T by applying the velocity profile (3) and the following boundary conditions

$$\begin{aligned}\frac{\partial T}{\partial r}&=0 \quad \text{at} \quad r=0,\\ T&=T_s \quad \text{at} \quad r=r_s,\end{aligned} \qquad (6)$$

where T_s is the surface temperature.

[100 marks]

2. *Pertimbangkan suatu aliran lamina bendalir likat yang mantap dengan suhu* T *dalam tiub bulat berjejari* r_s *yang* x *diukur di sepanjang tiub dan* r *paksi jejarian yang berserenjang dengan* x. *Andaikan bendalir memasuki tiub dengan halaju seragam merentasi keratan rentas aliran. Persamaan momentum untuk aliran simetri sepaksi dalam tiub bulat pada arah* x *diberikan oleh*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u}{\partial r}\right)-\frac{dP}{dx}=0, \qquad (2)$$

dengan tekanan P *tidak bersandar dengan* r *dan kelikatan dinamik bendalir* μ *adalah malar.*

(a) *Terbitkan profil halaju terbangun penuh berikut*

$$u=\frac{r_s^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1-\frac{r^2}{r_s^2}\right), \qquad (3)$$

dengan menggunakan persamaan momentum (2) tertakluk kepada syarat-syarat sempadan

$$\frac{\partial u}{\partial r}=0 \quad \text{pada} \quad r=0, \qquad u=0 \quad \text{pada} \quad r=r_s. \qquad (4)$$

(b) *Anggapkan bahawa persamaan tenaga dengan kadar haba malar untuk aliran simetri sepaksi dalam tiub bulat ialah*

$$u\left(\frac{dT_m}{dx}\right)=\alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \qquad (5)$$

dengan T_m *dan* α *masing-masing adalah suhu jisim-purata dan resapan terma bendalir malar. Terbitkan profil suhu terbangun penuh* T *dengan mengaplikasikan profil halaju (3) dan syarat-syarat sempadan berikut*

$$\frac{\partial T}{\partial r}=0 \quad \text{at} \quad r=0, \qquad T=T_s \quad \text{at} \quad r=r_s, \qquad (6)$$

dengan T_s *adalah suhu permukaan.*

[100 markah]

3. Consider constant-property flow along a surface with constant free-stream velocity. Let the temperature difference between the surface and the fluid, $T_s - T_\infty$, vary as x^m, where m is a constant. This problem is a Blasius type solution to the flat plate boundary layer with constant free stream velocity and variable surface temperature. The governing equation consists of the energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{7}$$

with boundary conditions

$$\begin{aligned} T(0,y) &= T_\infty, \\ T(x,0) &= T_s, \\ T(x,\infty) &= T_\infty. \end{aligned} \tag{8}$$

By introducing the following nondimensional temperature

$$\theta = \frac{T - T_\infty}{T_s - T_\infty} = \frac{T - T_\infty}{\phi(x)} = \frac{T - T_\infty}{Cx^m}, \tag{9}$$

show that a similarity solution to the energy equation is obtainable under these conditions. Carry out the necessary calculations to obtain the Nusselt number as a function of Reynolds number for $\Pr = 0.7$ (air) and $m = 1$.

(Note: Take $\xi = x, \psi = \sqrt{\upsilon x u_\infty}\, f$ and $\eta = y / \sqrt{\upsilon x / u_\infty}$ where υ is kinematic viscosity. ψ is the stream function, which is defined in a usual way as $u = \partial\psi / \partial y$ and $v = -\partial\psi / \partial x$)

[100 marks]

4. Consider a steady mixed convection boundary layer flow past a vertical semi-infinite flat plate embedded in a porous medium filled with a nanofluid. It is assumed that the free stream velocity and the ambient temperature (far flow from the plate) are U_∞ and T_∞, respectively. It is also assumed that the temperature of the plate is T_w, where $T_w > T_\infty$ corresponds to a heating plate (assisting flow) and $T_w < T_\infty$ corresponds to a cooling plate (opposing flows). The governing equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{10}$$

$$\frac{\mu_{nf}}{\mu_f}\, u = \frac{\mu_{nf}}{\mu_f} U_\infty + \frac{g\,K\left[\varphi\,\rho_s\,\beta_s + (1-\varphi)\,\rho_f\,\beta_f\right]}{\mu_f}\,(T - T_\infty), \tag{11}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2}, \tag{12}$$

3. *Pertimbangkan aliran bersifat malar di sepanjang permukaan dengan halaju strim bebas mlar. Andaikan perbezaan suhu di antara permukaan dan bendalir,* $T_s - T_\infty$*, berubah dengan* x^m*, yang m adalah pemalar. Masalah ini merupakan sejenis penyelesaian Blasius terhadap lapisan sempadan bagi plat rata dengan halaju strim bebas malar dan suhu permukaan boleh ubah. Persamaan menakluk yang terdiri daripada persamaan tenaga ialah*

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \qquad (7)$$

dengan syarat-syarat sempadan

$$\begin{aligned} T(0,y) &= T_\infty, \\ T(x,0) &= T_s, \\ T(x,\infty) &= T_\infty. \end{aligned} \qquad (8)$$

Dengan memperkenalkan suhu tak berdimensi berikut

$$\theta = \frac{T - T_\infty}{T_s - T_\infty} = \frac{T - T_\infty}{\phi(x)} = \frac{T - T_\infty}{Cx^m}, \qquad (9)$$

tunjukkan bahawa penyelesaian keserupaan bagi persamaan tenaga boleh diperoleh berdasarkan syarat-syarat yang diberikan. Lakukan pengiraan untuk mendapatkan nombor Nusselt dalam bentuk fungsi nombor Reynolds untuk $\Pr = 0.7$ *(udara) dan* $m = 1$.
(Nota: Ambil $\xi = x, \psi = \sqrt{\upsilon x u_\infty}\, f$ dan $\eta = y/\sqrt{\upsilon x / u_\infty}$ *dengan* υ *adalah kelikatan kinematik.* ψ *adalah fungsi strim, yang selalunya ditakrifkan sebagai* $u = \partial\psi/\partial y$ *dan* $v = -\partial\psi/\partial x$*)*

[100 markah]

4. *Pertimbangkan suatu aliran lapisan sempadan olakan campuran yang mantap terhadap plat rata menegak separuh tak terhingga dalam medium berongga yang diisi dengan bendalir nano. Andaikan halaju strim bebas dan suhu persekitaran (aliran jauh dari plat) masing-masing adalah* U_∞ *dan* T_∞*. Andaikan juga suhu plat ialah* T_w*, dengan* $T_w > T_\infty$ *mewakili plat yang dipanaskan (aliran membantu) dan* $T_w < T_\infty$ *mewakili plat yang disejukkan (aliran menentang). Persamaan-persamaan menakluk diberikan oleh*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (10)$$

$$\frac{\mu_{nf}}{\mu_f}u = \frac{\mu_{nf}}{\mu_f}U_\infty + \frac{gK\left[\varphi\rho_s\beta_s + (1-\varphi)\rho_f\beta_f\right]}{\mu_f}(T - T_\infty), \qquad (11)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2}, \qquad (12)$$

subject to the boundary conditions

$$v = 0, \quad T = T_w \quad \text{at} \quad y = 0,$$
$$u = U_\infty, \quad T = T_\infty \quad \text{at} \quad y = y_\infty. \qquad (13)$$

Here x and y are the Cartesian coordinates measured along the plate and normal to it, respectively, u and v are the velocity components along x and y axes, respectively, T is the temperature of the nanofluid, g is the acceleration due to gravity, φ is the nanoparticle volume fraction, μ_{f} is the dynamic viscosity of the base fluid, β_{f} and β_s are the coefficients of thermal expansion of the fluid and of the solid, ρ_{f} and ρ_{s} are the densities of the fluid and of the solid fractions, μ_{nf} is the viscosity of the nanofluid and α_{nf} is the thermal diffusivity of the nanofluid, which are given by

$$\mu_{\mathrm{nf}} = \frac{\mu_{\mathrm{f}}}{(1\text{-}\varphi)^{2.5}}, \quad \alpha_{\mathrm{nf}} = \frac{k_{\mathrm{nf}}}{(\rho \mathrm{C_p})_{\mathrm{nf}}}, \quad (\rho C_p)_{\mathrm{nf}} = (1-\varphi)(\rho C_p)_{\mathrm{f}} + \varphi(\rho C_p)_s,$$
$$\frac{k_{\mathrm{nf}}}{k_{\mathrm{f}}} = \frac{(k_{\mathrm{s}} + 2k_{\mathrm{f}}) - 2\varphi(k_{\mathrm{f}} - k_{\mathrm{s}})}{(k_{\mathrm{s}} + 2k_{\mathrm{f}}) + \varphi(k_{\mathrm{f}} - k_{\mathrm{s}})}. \qquad (14)$$

The viscosity of the nanofluid μ_{nf} can be approximated as viscosity of a base fluid μ_f containing dilute suspension of fine spherical particles, k_{nf} is the thermal conductivity of the nanofluid, k_f and k_s are the thermal conductivities of the base fluid and of the solid and $(\rho C_p)_{nf}$ is the heat capacitance of the nanofluid.
(Note: ψ is the stream function, which is defined in a usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$)

Show that the governing equations (10)-(12) can be reduced to the following system of ordinary differential equations

$$\frac{1}{(1-\varphi)^{2.5}} f' = \frac{1}{(1-\varphi)^{2.5}} + \left[1-\varphi+\varphi(\rho_s/\rho_f)(\beta_s/\beta_f)\right]\lambda\theta, \qquad (15)$$

$$\frac{k_{nf}/k_f}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\theta'' + f\theta' = 0, \qquad (16)$$

tertakluk kepada syarat-syarat sempadan

$$v = 0, \quad T = T_w \quad \text{pada} \quad y = 0,$$
$$u = U_\infty, \quad T = T_\infty \quad \text{pada} \quad y = y_\infty. \tag{13}$$

x dan y masing-masing adalah koordinat-koordinat Cartesian yang diukur di sepanjang plat dan berserenjang dengannya, u dan v masing-masing adalah komponen-komponen halaju pada paksi x dan y, T ialah suhu bendalir nano, g ialah pecutan graviti, φ ialah pecahan isipadu zarah nano, μ_{f} ialah kelikatan dinamik bendalir asas, β_{f} dan β_s masing-masing adalah pekali-pekali pengembangan terma bendalir dan pepejal (zarah), ρ_{f} dan ρ_{s} masing-masing adalah ketumpatan bendalir dan pecahan-pecahan pepejal, μ_{nf} ialah kelikatan bendalir nano dan α_{nf} ialah resapan terma bendalir nano, yang diberikan oleh

$$\mu_{\mathrm{nf}} = \frac{\mu_{\mathrm{f}}}{(1\text{-}\varphi)^{2.5}}, \quad \alpha_{\mathrm{nf}} = \frac{k_{\mathrm{nf}}}{(\rho \mathrm{C_p})_{\mathrm{nf}}}, \quad (\rho C_p)_{\mathrm{nf}} = (1-\varphi)(\rho C_p)_{\mathrm{f}} + \varphi(\rho C_p)_s,$$
$$\frac{k_{\mathrm{nf}}}{k_{\mathrm{f}}} = \frac{(k_{\mathrm{s}} + 2k_{\mathrm{f}}) - 2\varphi(k_{\mathrm{f}} - k_{\mathrm{s}})}{(k_{\mathrm{s}} + 2k_{\mathrm{f}}) + \varphi(k_{\mathrm{f}} - k_{\mathrm{s}})}. \tag{14}$$

Kelikatan bendalir nano μ_{nf} adalah hampiran kepada kelikatan bendalir asas μ_f yang mengandungi ampaian cair zarah-zarah sfera halus, k_{nf} ialah kekonduksian terma bendalir nano, k_f dan k_s masing-masing adalah kekonduksian terma bendalir asas dan pepejal, dan $(\rho C_p)_{nf}$ ialah kemuatan haba bendalir nano. the heat capacitance of the fluid nanofluid. (Nota: ψ adalah fungsi strim, yang selalunya ditakrifkan sebagai $u = \partial\psi / \partial y$ dan $v = -\partial\psi / \partial x$)

Tunjukkan bahawa persamaan-persamaan menakluk (10)-(12) terturun kepada sistem persamaan pembezaan biasa seperti berikut

$$\frac{1}{(1-\varphi)^{2.5}} f' = \frac{1}{(1-\varphi)^{2.5}} + \left[1 - \varphi + \varphi(\rho_s / \rho_f)(\beta_s / \beta_f)\right]\lambda\theta, \tag{15}$$

$$\frac{k_{nf} / k_f}{(1-\varphi) + \varphi(\rho C_p)_s / (\rho C_p)_f} \theta'' + f\theta' = 0, \tag{16}$$

with the boundary conditions (13) becoming

$$f(0)=0, \quad \theta(0)=1, \\ f'(\eta_\infty)=1, \quad \theta(\eta_\infty)=0, \tag{17}$$

by using similarity variables of the following form

$$\psi=\alpha_f\,(2\,Pe_x)^{1/2}\; f(\eta), \quad \theta(\eta)=(T-T_\infty)/(T_w-T_\infty), \quad \eta=Pe_x^{1/2}\; y/(x\sqrt{2}). \tag{18}$$

Here primes denote differentiation with respect to η and λ is the constant mixed convection parameter, which is defined as

$$\lambda=\frac{Ra_x}{Pe_x}, \tag{19}$$

with $Ra_x=\rho_f\; g\,K\,\beta_f\,(T_w-T_\infty)\,x/(\mu_f\;\alpha_f)$ being the local Rayleigh number for a porous medium. It is worth mentioning that $\lambda>0$ corresponds to an assisting flow (heated plate), $\lambda<0$ corresponds to opposing flows (cooled plate) and $\lambda=0$ corresponds to the forced convection flow.

[150 marks]

dengan syarat-syarat sempadan (13) menjadi

$$f(0) = 0, \quad \theta(0) = 1,$$
$$f'(\eta_\infty) = 1, \quad \theta(\eta_\infty) = 0, \qquad (17)$$

dengan menggunakan pemboleh-pemboleh ubah keserupaan berikut

$$\psi = \alpha_f \,(2Pe_x)^{1/2} \, f(\eta), \quad \theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \quad \eta = Pe_x^{1/2} \, y/(x\sqrt{2}). \qquad (18)$$

Tandaan $'$ *mewakili terbitan terhadap* η *dan* λ *ialah parameter olakan campuran malar, yang didefinisikan sebagai*

$$\lambda = \frac{Ra_x}{Pe_x}, \qquad (19)$$

dengan $Ra_x = \rho_f \, g \, K \, \beta_f \,(T_w - T_\infty)\, x/(\mu_f \, \alpha_f)$ *merupakan nombor Rayleigh setempat bagi medium berongga. Perlu dinyatakan di sini bahawa* $\lambda > 0$ *mewakili aliran membantu (plat panas),* $\lambda < 0$ *mewakili aliran menentang (plat sejuk) dan* $\lambda = 0$ *mewakili aliran olakan paksa.*

[150 markah]

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